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Newton-like method for singular 2-regular system of nonlinear equations

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction

Let $F:D\subset \mathbb{R}^n\to\mathbb{R}^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^*\in D$ of the equation

$$F(x) = 0. (1)$$

Definition 1

A linear operator $\Psi_2(h): R^n \to R^m$, $h \in R^n$ is called 2-factoroperator, if $\Psi_2(h) = F'(x^*) + P^{\perp}F'(x^*)h$, (2)

where

 P^{\perp} - denotes the orthogonal projection on $(\operatorname{Im} F'(x))^{\perp}$ in $R^{n}[1]$.

Definition 2

Operator F is called 2-regular in x^* on the element $h \in \mathbb{R}^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$\operatorname{Im}\Psi_{2}(h)=R^{m}.$$

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Definition 3

Operator F is called 2-regular in x^* , if F is 2-regular on the set $K_2(x^*)\setminus\{0\}$, where

$$K_2(x^*) = KerF'(x^*) \cap Ker^2 P^{\perp} F''(x^*), \tag{3}$$

$$Ker^2P^{\perp}F^{"}(x^*) = \{h \in R^n : P^{\perp}F^{"}(x^*)[h]^2 = 0\}.$$

We need the following assumption on F:

A1) completely degenerated in x*:

$$\operatorname{Im} F'(x^*) = 0. (4)$$

A2) operator F is 2-regular in x*:

Im
$$F'(x^*)h = R^m$$
 for $h \in K_2(x^*), h \neq 0.$ (5)

$$KerF^{*}\left(x^{*}\right)\neq\left\{ 0\right\} .\tag{6}$$

If F satisfies A1 in x*, then

$$K_2(x^*) = Ker^2 F'(x^*) = \{ h \in R^n : F'(x^*)[h]^2 = 0 \}.$$
 (7)

In [1] it was proved, that if n=m, then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}'(x_k) + P_k^{\perp} F'(x_k) h_k \right\}^{-1} \cdot \left\{ F(x_k) + P_k^{\perp} F'(x_k) h_k \right\}, \tag{8}$$

where

 P_k^{\perp} - denotes orthogonal projection on $\left(\operatorname{Im} \hat{F}'(x_k)\right)^{\perp}$ in R^n ,

$$h_k \in Ker\hat{F}'(x_k), \|h_k\| = 1$$

converges Q-quadratically to x*.

The matrices $\hat{F}'(x_k)$ obtained from $F'(x_k)$ by replacing all elements, whose absolute values do not increase v>0, by zero, where $\mathbf{n} = \mathbf{n}_k = \|F(x_k)\|^{(1-a)/2}$, $0 < \alpha < 1$.

In the case n = m+1 the operator

$$\left\{\hat{F}'(x_k) + P_k^{\perp} F''(x_k) h_k\right\}^{-1}$$

in method (8) is replaced by the operator

$$\left[\hat{F}'(x_k) + P_k^{\perp} F''(x_k) h_k\right]^{+} \tag{9}$$

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined (n>m) and degenerated in x^* .

2. Extending of the system of equation

Now we construct the operator $\Phi: \mathbb{R}^n \to \mathbb{R}^{n-1}$ with the properties (4), (5) and such that $\Phi(\mathbf{x}^*)=0$ [2].

Assume

A4) Let $F(x)=[f_1(x), f_2(x), ..., f_m(x)]^T$, n>m is two continuously differentiable in some neighbourhood $U \subset \mathbb{R}^n$ of the point x^* .

Denote:

H=lin{h} for
$$h \in Ker^2F^*(x^*)$$
, h $\neq 0$.

 $P = P_{H^{\perp}}$ denotes the orthogonal projection R^n on H^{\perp}

$$f_i(x) = P(f_i(x))^T$$
 for i=1,2,...,m.

For each system of indices $i_1, i_2, ..., i_{n-m-1} \subset \{1, 2, ..., m\}$ and vectors $h_1, h_2, ..., h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x)h \\ j(x) \end{bmatrix},\tag{10}$$

where

$$j(x): R^{n} \to R^{r}, \quad \text{r=n-m-1},$$

$$j(x) = PF'(x)h^{n}, \quad h^{n} = [h_{1}, h_{2}, ..., h_{r}]^{T},$$

$$j(x) = \begin{bmatrix} f_{i_{1}}^{n}(x)h_{1} \\ \mathbf{M} \\ f_{i_{r}}^{n}(x)h_{r} \end{bmatrix}. \tag{11}$$

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[\Phi'(x_k)\right]^+ \cdot \Phi(x_k), \quad k=0,1,2,....$$
 (12)

quadraticaly converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left\{ B_k \right\}^+ \cdot \Phi(x_k). \tag{13}$$

The operator $\Phi^{'}$ will by approximated by matrices $\{B_k\}.$ Let

$$S_k = X_{k+1} - X_k \,. \tag{14}$$

We propose matrices B_k which satisfy the secant equation:

$$B_{k+1}S_k = \Phi(x_{k+1}) - \Phi(x_k)$$
 for k=0,1,2,... (15)

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:

$$B_{k+1} = B_k - \frac{r_k S_k^T}{S_k^T S_k}$$
 for k=0,1,2,... (16)

where

$$r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k.$$
 (17)

We will prove for this method:

Q-linear convergence to x^* i.e. there exists $q \in (0,1)$ such, that

$$||x_{k+1} - x^*|| \le q ||x_k - x^*||$$
 for $k = 0, 1, 2, ...$ (18)

and next Q-superlinear convergence to x*, i.e.:

$$\lim_{k \to \infty} \frac{\left\| x_{k+1} - x^* \right\|}{\left\| x_k - x^* \right\|} = 0.$$
 (19)

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator $F'(x^*)$ is nonsingular.

Theorem 1 (The Bounded Deterioration Theorem)

Let F satisfies the assumptions A1-A4. If exist constants $q_1 \ge 0$ and $q_2 \ge 0$ such that matrices $\{B_k\}$ satisfy the inequality:

$$\|B_{k+1} - \Phi'(x^*)\| \le (1 + q_1 r_k) \|B_k - \Phi'(x^*)\| + q_2 r_k,$$
 (20)

then there are constants e > 0 i d > 0 such, that if

$$\|\mathbf{x}_0 - \mathbf{x}^*\| \le e$$
 and $\|\mathbf{B}_0 - \Phi'(\mathbf{x}^*)\| \le d$,

then the sequence

$$x_{k+1} = x_k - B_k^+ \Phi(x_k)$$

converges Q-linearly to x*.

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

Theorem 2 (Linear convergence)

Let F satisfies the assmuptions A1-A4. Then the method

$$x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi(x_k),$$

$$B_{k+1} = B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k}$$

locally and Q-linearly converges to x*.

Proof

To prove the Theorem we should prove the inequality (20) from Theorem 1. Now we notice:

$$\begin{split} & \left\| B_{k+1} - \Phi^{'}(x^{*}) \right\| = \left\| B_{k} - \frac{\left\{ \Phi(x_{k+1}) - \Phi(x_{k}) - B_{k}s_{k} \right\} s_{k}^{T}}{s_{k}^{T} s_{k}} - \Phi^{'}(x^{*}) \right\| \leq \\ & \leq \left\| B_{k} - \Phi^{'}(x^{*}) \right\| + \left\| \frac{\left\{ \Phi(x_{k+1}) - \Phi(x_{k}) - B_{k}s_{k} \right\} s_{k}^{T}}{s_{k}^{T} s_{k}} \right\| \leq \left\| B_{k} - \Phi^{'}(x^{*}) \right\| + \\ & + \left\| \frac{\left\{ \Phi(x_{k+1}) - \Phi(x_{k}) - \Phi^{'}(x^{*}) s_{k} + \Phi^{'}(x^{*}) s_{k} - B_{k}s_{k} \right\} s_{k}^{T}}{s_{k}^{T} s_{k}} \right\| \leq \left\| B_{k} - \Phi^{'}(x^{*}) \right\| + \\ & + \left\| \frac{\left(\Phi(x_{k+1}) - \Phi^{'}(x^{*}) (x_{k+1} - x^{*}) \right) s_{k}^{T}}{s_{k}^{T} s_{k}} \right\| + \left\| \frac{\left(\Phi(x_{k}) - \Phi^{'}(x^{*}) (x_{k} - x^{*}) \right) s_{k}^{T}}{s_{k}^{T} s_{k}} \right\| + \\ & + \left\| \frac{\left(\Phi^{'}(x^{*}) - B_{k} \right) s_{k} s_{k}^{T}}{s_{k}^{T} s_{k}} \right\| \leq \left\| \Phi^{'}(x^{*}) - B_{k} \right\| \left(1 + q_{1}r_{k} \right) + c_{1} \frac{\left\| x_{k+1} - x^{*} \right\|^{2} \left\| s_{k} \right\|}{\left\| s_{k}^{T} s_{k} \right\|} + \\ & + c_{2} \frac{\left\| x_{k} - x^{*} \right\|^{2} \left\| s_{k} \right\|}{\left\| s_{k}^{T} s_{k} \right\|} \leq \left\| \Phi^{'}(x^{*}) - B_{k} \right\| \left(1 + q_{1}r_{k} \right) + q_{2}r_{k}, \end{split}$$

where $c_1 > 0$, $c_2 > 0$, $q_1 > 0$, $q_2 > 0$, $r_k = \max\{\|x_{k+1} - x^*\|, \|x_k - x^*\|\}$.

Theorem 3 (Q-superlinear convergence)

Let F satisfies the assmuptions A1-A4 and the sequence

$$x_{k+1} = x_k - \{B_k\}^{-1} \cdot \Phi(x_k),$$

$$B_{k+1} = B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k}$$

linearly converges to \mathbf{x}^* . Then the sequence $\{x_k\}$ Q-superlinearly converges to \mathbf{x}^* .

Proof.

Matrices B_k satisfy secant equation (15), so

$$B_{k+1} = P_{L_k}^{\perp} B_k \tag{21}$$

where

$$L_{k} = \left\{ X : X s_{k} = y_{k}, \text{ where } y_{k} = \Phi'(x_{k+1}) - \Phi'(x_{k}) \right\}$$
 (22)

Denote

$$H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) dt$$

We have $H_k \in L_k$ [4].

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From (21) and [3] it follows:

$$\|B_{k+1} - B_k\|^2 + \|B_{k+1} - H_k\|^2 = \|B_k - H_k\|^2$$
, for $i = 0, 1, 2, ...$

By lemma 2 [5] we get $\sum_{k=1}^{\infty} \|B_{k+1} - B_k\|^2 < \infty$, thus we obtain

$$\parallel B_{k+1} - B_k \parallel \rightarrow 0.$$

This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. \Box

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of $F^{"}(x_k)$.

References

- [1] Izmailov A.F., Tretiakov A.A., *Factor-analysis of non-linear mappings*, Moscow, Nauka, Fizmatlit Publishing Company, (1994) 336.
- [2] Chmura A., Tretiakov A.A., New Methods for the solution of nonlinear equations with singularities-Nonlinear Oscillations, 2 (1999) 1.
- [3] Broyden C.G., J.E. Dennis, J.J. More, On the local and superlinear convergence of quasi-Newton methods, Journal Inst. Math. Appl., 12 (1973) 223.
- [4] Ortega J.M., Reinboldt W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).
- [5] Grzegórski S.M., Rescaled multilevel least-change almost secant methods, "Matematyka Stosowana" XXXIII, (1991).
- [6] Grzegórski S.M., Orthogonal projections on convex sets for Newton-like methods, SIAM J.Numer. Anal., 22 (1985) 1208.