



Two hierarchies of \mathbf{R} -recursive functions

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Abstract

In the paper some aspects of complexity of \mathbf{R} -recursive functions are considered. The limit hierarchy of \mathbf{R} -recursive functions is introduced by the analogy to the μ -hierarchy. Then its properties and relations to the μ -hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorzczuk hierarchy, compare [1]).

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorzczuk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals \mathbf{R} (called \mathbf{R} -recursive functions) in the analogous way to the classical recursive functions on the natural numbers \mathbf{N} . His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation μ , which is used to construct μ -hierarchy of \mathbf{R} -recursive functions.

It was shown [5] that the zero-finding operator μ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to μ -hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called **R**-recursive functions [4].

Definition 2.1 *The set of **R**-recursive functions is generated from the constants 0,1 by the operations:*

- 1) *composition: $h(\bar{x}) = f(g(\bar{x}))$;*
- 2) *differential recursion: $h(\bar{x}, 0) = f(\bar{x}), \partial_y h(\bar{x}, y) = g(\bar{x}, y, h(\bar{x}, y))$ (the equivalent formulation can be given by integrals: $h(\bar{x}, y) = f(\bar{x}) + \int_0^y g(\bar{x}, y', h(\bar{x}, y')) dy'$);*
- 3) *m-recursion $h(\bar{x}) = m_y f(\bar{x}, y) = \inf \{y : f(\bar{x}, y) = 0\}$, where infimum chooses the number **y** with the smallest absolute value and for two **y** with the same absolute value the negative one;*
- 4) *vector-valued functions can be defined by defining their components.*

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if **h** is defined by a differential recursion then **h** is defined only where a finite and unique solution exists. This is why the set of **R**-recursive functions includes also partial functions. We use (after [4]) the name of **R**-recursive functions in the article, however we should remember that in reality we have partiality here (partial **R**-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive **y** or just below some negative **y** then the infimum operation returns that **y** even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] *m*-operation is replaced by infinite limits: $h(\bar{x}) = \liminf_{y \rightarrow \infty} g(\bar{x}, y)$, $h(\bar{x}) = \limsup_{y \rightarrow \infty} g(\bar{x}, y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$, which can be in the obvious way obtained from limsup, liminf:

Corollary 2.2 *The class of \mathbf{R} -recursive functions is closed under the operations of infinite limits: $h(\bar{x}) = \liminf_{y \rightarrow \infty} g(\bar{x}, y)$, $h(\bar{x}) = \limsup_{y \rightarrow \infty} g(\bar{x}, y)$, $h(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$.*

3. Hierarchies

The operator m is a key operator in generating the \mathbf{R} -recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of m in the definition of a given f .

Definition 3.1 ([4]) *For a given \mathbf{R} -recursive expression $s(\bar{x})$, let $M_{x_i}(s)$ (the m -number with respect to x_i) be defined as follows:*

$$M_x(0) = M_x(1) = M_x(-1) = 0, \quad (1)$$

$$M_x(f(g_1, g_2, \dots)) = \max_j (M_{x_j}(f) + M_x(g_j)), \quad (2)$$

$$M_x\left(h = f + \int_0^y g(\bar{x}, y', h) dy'\right) = \max(M_x(f), M_x(g), M_h(g)), \quad (3)$$

$$M_y\left(h = f + \int_0^y g(\bar{x}, y', h) dy'\right) = \max(M_{y'}(g), M_h(g)), \quad (4)$$

$$M_x(m_y f(\bar{x}, y)) = \max(M_x(f), M_y(f)) + 1, \quad (5)$$

where x can be any x_1, \dots, x_n for $\bar{x} = (x_1, \dots, x_n)$.

For an \mathbf{R} -recursive function f , let $M(f) = \max_{x_i}(s)$ minimized over all expressions s that define f . Now we are ready to define M -hierarchy (m -hierachy) as a family of $M_j = \{f : M'(f) \leq j\}$.

Let us construct the analogous definition of L -hierarchy by replacing in the above definition M_x by L_x and changing line (5) to the following form (5'):

$$\begin{aligned} L_x\left(\liminf_{y \rightarrow \infty} g(\bar{x}, y)\right) &= L_x\left(\limsup_{y \rightarrow \infty} g(\bar{x}, y)\right) = \\ &= L_x\left(\lim_{y \rightarrow \infty} g(\bar{x}, y)\right) = \max(L_x(f), L_y(f)) + 1. \end{aligned}$$

For an \mathbf{R} -recursive function f , let $L(f) = \max_i L_{x_i}(s)$ minimized over all expressions s that define f without using the m -operation.

Definition 3.2 The **L**-hierarchy is a family of $L_j = \{f : L(f) \leq j\}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function f .

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 The classes M_0 and M_1 are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive **R**-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, xy , x/y , e^x , $\ln x$, y^x , $\sin x$, $\cos x$ are primitive **R**-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. The Kronecker **d** function, the signum function and absolute value belong to the first level (L_1) of limit hierarchy.

Proof. It is sufficient to take the following definitions [5]: hence $d(0) = 1$ and

for all $x \neq 0$ we have $d(x) = 0$ let us define $d(x) = \liminf_{y \rightarrow \infty} \left(\frac{1}{1+x^2} \right)^y$. Now

from the expression $\liminf_{y \rightarrow \infty} \arctan xy = \begin{cases} p/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -p/2, & \text{if } x < 0, \end{cases}$ we obtain

$$\operatorname{sgn}(x) = \frac{\liminf_{y \rightarrow \infty} \arctan xy}{2 \arctan 1} \text{ and } |x| = \operatorname{sgn}(x)x.$$

We should be careful with definitions of functions by cases:

Lemma 3.5 For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \mathbf{M} & \mathbf{M} \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k-1 \end{cases}$ and $g_i \in L_{n_i}$ for all $1 \leq i \leq k$,

$f \in L_m$ the function h belongs to $L_{\max(n_1, \dots, n_k, m+1)}$

Proof. Let us see that $eq(x, y) = d(x - y) \in L_1$ and

$$ge(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1. \quad \text{Then of course}$$

$$h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x}) eq(f(\bar{x}), i-1) + g_k(\bar{x}) ge(f(\bar{x}), k-1). \quad \square$$

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 *The function $\Theta(x)$ (equal to 1 if $x \geq 0$, otherwise 0), maximum $\max(x, y)$, square-wave function s are in L_2 , the function $p(x)$ such that $p(x) = 1$ for $x \in [2n, 2n+1]$ and $p(x) = 0$ for $x \in [2n+1, 2n+2]$ is in L_2 and the floor function $\lfloor x \rfloor$ is in L_3 .*

Proof. We give the proper definitions (from [6]) for these functions. Let

$$\Theta(x) = d(x - |x|),$$

$$\max(x, y) = xd(x - y) + (1 - d(x - y))[x\Theta(x - y) + y\Theta(y - x)],$$

$$s(x) = \Theta(\sin(px)).$$

The function $p(x)$ can be given as $s(x) \left(1 - d \left(\sin \frac{(x-1)p}{2} \right) \right)$, so $p \in L_2$.

The floor function we can define by the auxiliary function $w(0) = 0$, $\partial_x w(x) = 2\Theta(-\sin(2px))$ as

$$\lfloor x \rfloor = \begin{cases} 2w(x/2) & \text{if } p(x) = 1, \\ 2w((x-1)/2) & \text{if } p(x) = 0. \end{cases}$$

From the above equation we have $\lfloor x \rfloor$ in L_3 . \square

Let us recall that if $f : R^n \rightarrow R$ is an \mathbf{R} -recursive function then the function $f_{iter}(i, \bar{x})$ is \mathbf{R} -recursive, too.

Lemma 3.7 *Let $f : R^n \rightarrow R$ belongs to the class L_i , then we have $f_{iter} : R^{n+1} \rightarrow R$ is in $L_{\max(2, i)}$.*

Proof. The definitions, which were given by Moore [3] $f_{iter}(i, \bar{x}) = h(2i)$, where

$$h(0) = g(0) = \bar{x},$$

$$\begin{aligned} \partial_t g(t) &= [f(h(t)) - h(t)]s(t), \\ \partial_t h(t) &\geq \left[\frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \end{aligned}$$

with s - a square wave function in L_2 and $r(0) = 0$, $\partial_t r(t) = 2s(t) - 1$, $r, s \in L_2$ give us the desirable statement. \square

Lemma 3.8 *The \mathbf{R}^1 -recursive functions $g_2 : \mathbf{R}^2 \rightarrow \mathbf{R}$, $g_2^1, g_2^2 : \mathbf{R} \rightarrow \mathbf{R}$ such that $(\forall x, y \in \mathbf{R}) g_2^1(g_2(x, y)) = x$, $(\forall x, y \in \mathbf{R}) g_2^2(g_2(x, y)) = y$, have the following properties: g_2, g_2^1 are in L_{10} , g_2^2 is in L_{14} .*

Proof. We have the auxiliary functions $\Gamma_2, \Gamma_2^1, \Gamma_2^2$, which are coding and decoding functions in the interval $(0, 1) : \Gamma_2(x, y) = c(x) + c(y)/10$, where

$$c(x) = \lim_{i \rightarrow \infty} z(a(i, x))/10^{2i} + b(i, x)/10^i,$$

and later $z(x) = \lim_{i \rightarrow \infty} z'_{iter}(i, x)$,

$$\begin{aligned} z'_{iter}(i, a_1 \dots a_n . a_{n+1} \dots) &= a_1 \dots a_n 0 \dots a_{n+1} 0 . a_{n+i+1} \dots, \\ a(i, 0 . a_1 a_2 \dots a_i \dots) &= 0 . a_1 \dots a_i \\ b(i, 0 . a_1 a_2 \dots a_i \dots) &= 0 . 0 \dots 0 a_{i+1} \dots, \end{aligned}$$

$(z'(x) = \begin{cases} 100 \lfloor x \rfloor + 10(x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x; \end{cases} \in L_4, a, b \in L_4$. Also z'_{iter} belongs

to L_4 , hence $\Gamma_2(x, y) \in L_{10}$, decoding of the first element is described in the symmetric way so $\Gamma_2^1(x)$ is in L_{10} , but $\Gamma_2^2(x) = \Gamma_2^1(10 - \lfloor 10x \rfloor)$ so $\Gamma_2^2 \in L_{14}$.

The functions $\Gamma_2, \Gamma_2^1, \Gamma_2^2$ can be extended to all reals by one-to-one $f : (0, 1) \rightarrow \mathbf{R} \in L_0$ without the loss of their class. \square

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions $g_n : \mathbf{R}^n \rightarrow \mathbf{R}$ and $g_n^i : \mathbf{R} \rightarrow \mathbf{R}$ for $i = 1, \dots, n$ such that

$$(\forall i)(\forall x_1, \dots, x_n \in \mathbf{R}) g_n^i(g_n(x_1, \dots, x_n)) = x_i$$

in the same class: $g_n, g_n^1 \in L_{10}$ and $(\forall i > 1) g_n^i \in L_{14}$.

We finish this part with the important form of defining: a new function is given as a product of values \mathbf{f} in some integer points.

Lemma 3.9 *There exists such constant $p \in \mathbb{N}$ that for the function*

$$\prod_{z=0}^y f(\bar{x}, z) = \begin{cases} f(\bar{x}, 0) f(\bar{x}, 1) \dots f(\bar{x}, \lfloor y-1 \rfloor), & \text{if } y \geq 1, \\ 1, & \text{if } 0 \leq y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

if the function f is in the class L_m then $\prod_{z=0}^y f(\bar{x}, z)$ is in the class L_{m+p} (p is independent of m).

Proof. By the definitions

$$t(w) = g_{n+2} \left(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w) + 1, f \left(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w) \right) \cdot g_{n+2}^{n+2}(w) \right)$$

and

$$S(\bar{x}, z) = t_{\lfloor z \rfloor} \left(t_{\lfloor z \rfloor} \left(s(\bar{x}, 0) \right) \dots \right) = t_{iter} \left(\lfloor z \rfloor, g_{n+2}(\bar{x}, 0, 1) \right)$$

we get the property

$$\prod_{y=0}^z f(\bar{x}, y) = g_{n+2}^{n+2} \left(S(\bar{x}, z) \right).$$

From the definition of the limit hierarchy we get $\prod_{y=0}^z f(\bar{x}, y) \in L_{m+38}$. \square

In the rest of the paper we will use the constant p as the number of limits used in the recursive definition of the product $\prod_{y=0}^z f(\bar{x}, y)$ instead of the value 38.

The above constructions are tedious and can be improved with a better approximation of p .

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between L-hierarchy and M-hierarchy.

Theorem 4.1 *Let $f : R^n \rightarrow R$ be an \mathbf{R} -recursive function. Then if $f \in L_i$ then $f \in M_{10i}$.*

Proof. We use a simple induction here. The case $i=0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i=n$. Let $f \in L_{n+1}$ be defined as $f(\bar{x}) = \lim_{y \rightarrow \infty} g(\bar{x}, y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \mathbf{f} from g it is necessary to use at

most 10 m -operation. Hence for $g \in M_{10n}$ the function f satisfies $f \in M_{10n+10}$. Similar inferences hold for \liminf , \limsup . \square

Now we can give the result about the 'limit complexity' of the infimum operator m

Lemma 4.2 *If $f(\bar{x}, y): R^{n+1} \rightarrow R$ is in the class L_m then the function $g: R^n \rightarrow R$, $g(\bar{x}) = m_y f(\bar{x}, y)$ is in the class L_{m+3p+9} is from Lemma 3.9.*

Proof. Here we must employ the results from [6]. There we defined the function $g: R^n \rightarrow R$, $g(\bar{x}) = m_y f(\bar{x}, y)$ for $f(\bar{x}, y): R^{n+1} \rightarrow R$ (f - \mathbf{R} -recursive) replacing the m -operator by limit operation. First we introduced the function

$$Z^f(\bar{x}, z) = \begin{cases} \inf_y \{f: K^f(\bar{x}, y) = 0\}, & \text{if } z = 0 \text{ and } \exists y K^f(\bar{x}, y) = 0, \\ \text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(\bar{x}, y) \neq 0, \\ 1 & \text{if } z \neq 0, \end{cases}$$

given in the following way:

$$Z^f(\bar{x}, z) = \begin{cases} \text{undefined} & \text{if } (z = 0) \wedge (S^f(\bar{x}) < 1/12), \\ \sqrt{S^f(\bar{x}) - 1/12}, & \text{if } (z = 0) \wedge (S^f(\bar{x}) \geq 1/12) \\ & \wedge f(\bar{x}, \sqrt{S^f(\bar{x}) - 1/12}) = 0, \\ -\sqrt{S^f(\bar{x}) - 1/12}, & \text{if } (z = 0) \wedge (S^f(\bar{x}) \geq 1/12) \\ & \wedge f(\bar{x}, -\sqrt{S^f(\bar{x}) - 1/12}) = 0, \\ 1, & \text{if } z \neq 0. \end{cases}$$

where $S^f(\bar{x}) = \lim_{t \rightarrow \infty} S_1^f(\bar{x}, t) + \lim_{t \rightarrow \infty} S_2^f(\bar{x}, t)$. Both functions S_1^f , S_2^f are defined by an integration

$$S_i^f(\bar{x}, t) = \int y^2 \left(1 - h^f(\bar{x}, (-1)^{i+1} y - 1/2, (-1)^{i+1} y + 1/2) \right) dy, \quad i = 1, 2$$

from $h^f(\bar{x}, a, b) = \liminf_{t \rightarrow \infty} \prod_{w=0}^{z+1} K^f\left(\bar{x}, a + w \frac{b-a}{z}\right)$ where K^f is the characteristic function of f .

Hence we can conclude that if K^f is in the L_s then Z_f is in the class L_{s+p+3} . Let us finish with the definition of the characteristic function of the infimum of zeros of f (see Theorem 4.2 from [5])

$$K_m^f(y) = 1 - \lim_{a \rightarrow -\infty} \lim_{b \rightarrow \infty} \lim_{z \rightarrow \infty} G^f(\bar{x}, z, a, b, y),$$

where $G^f(\bar{x}, z, a, b, y)$ divides the interval $[a, b]$ into $2^{\lfloor z \rfloor}$ equal subintervals and gives the value 1 for y from the subintervals, which contains the least zero of f in $[a, b]$ and value 0 otherwise. Precisely for y from $\left[a, a + \frac{b-a}{2^{\lfloor z \rfloor}} \right]$

$$G^f(\bar{x}, z, a, b, y) = \begin{cases} 1, & \text{if } h^f\left(\bar{x}, a, a + \frac{b-a}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

for $y \in \left(a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}} \right)$ (where $k = 2, 3, \dots, 2^n$) we have:

$$G^f(\bar{x}, z, a, b, y) = \begin{cases} 1, & \text{if } \prod_{i=1}^{k-1} h^f\left(\bar{x}, a + \frac{(i-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{i(b-a)}{2^{\lfloor z \rfloor}}\right) \neq 0 \\ \wedge h^f\left(\bar{x}, a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

and for $Y \notin [A, B]$ the function g_x^f is equal to 2.

The definition of G_f is given by the cases with respect to the value of the expression given by $\prod h^f$, since for $f \in L_m$, the function $h_f \in L_{m+p+2}$ and $G^f \in L_{m+2p+3}$. Then we have $K_m^f \in L_{m+2p+6}$. Now we must use the function K_m^f in the same way as K^f which gives us Z_f in the class L_{m+3p+9} . The final definition of $g(\bar{x}) = m_y f(\bar{x}, y)$ ([5] Theorem 4.3) given below

$$g(\bar{x}) = \begin{cases} Z^{f^+}(\bar{x}, 0) - Z^{f^-}(\bar{x}, 0), & \text{if } S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12}, \\ Z^{f^+}(\bar{x}, 0), & \text{if } \left(S^{f^+}(\bar{x}) \geq \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12} \right) \\ & \text{or} \\ & \left(S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12} \right) \\ & \wedge Z^{f^+}(\bar{x}, 0) < Z^{f^-}(\bar{x}, 0), \\ -Z^{f^-}(\bar{x}, 0), & \text{if } \left(S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) \geq \frac{1}{12} \right) \\ & \text{or} \\ & \left(S^{f^+}(\bar{x}) < \frac{1}{12} \wedge S^{f^-}(\bar{x}) < \frac{1}{12} \right) \\ & \wedge Z^{f^+}(\bar{x}, 0) \geq Z^{f^-}(\bar{x}, 0), \end{cases}$$

where $f^+(\bar{x}, y) = \begin{cases} f(\bar{x}, y), & y \geq 0, \\ 1, & y < 0; \end{cases}$ $f^-(\bar{x}, y) = \begin{cases} f(\bar{x}, -y), & y > 0, \\ 1, & y \leq 0; \end{cases}$ remains the class of g identical to the class of Z^f , i.e. $g \in L_{m+3p+9}$. \square

Theorem 4.3 Let $f : R^n \rightarrow R$ be an \mathbf{R} -recursive function. Then for all $i \geq 0$ if $f \in M_i$ then $f \in L_{(3p+9)i}$.

The above statement is a simple consequence of the fact $M_0 = L_0$ and Lemma 4.2.

5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the m -operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a m -operator.

We also establish the proper relation between the levels of the limit hierarchy and m -hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also m -hierarchy) and Baire classes

[7]. Also the kind of a connection between the \sum_n^0 -measurable functions and \mathbf{R} -recursive functions is an open problem.

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