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JOANNA MARKOWICZ and STANISŁAW PRUS

Properties of modulus of monotonicity and Opial property in direct sums

Abstract. We give an example of a Banach lattice with a non-convex modulus of monotonicity, which disproves a claim made in the literature. Results on preservation of the non-strict Opial property and Opial property under passing to general direct sums of Banach spaces are established.

1. Introduction. Geometry of Banach spaces is an important field of functional analysis with many applications, in particular to metric fixed point theory. The most classical and most frequently applied geometric properties of Banach spaces are uniform convexity and uniform smoothness. There are scaling functions corresponding to these properties called the modulus of convexity and modulus of smoothness. It is well known that these properties are dual to each other. This theorem has its quantitative form in the so-called Lindenstrauss formula relating the modulus of smoothness of the dual space to the modulus of convexity of the initial space (see Proposition 1.e.2 [6]).

Most examples of Banach spaces are sequence spaces or function spaces. In such spaces we have the natural order which in many cases makes them Banach lattices. Having a lattice, we can consider not only general geometric properties but also specific properties related to order. The basic properties of this kind are uniform monotonicity and order uniform smoothness. They

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are in a sense lattice counterparts of uniform convexity and uniform smoothness. As in the classical case, they have corresponding scaling functions and in [5] a counterpart of the Lindenstrauss formula was given. However, [5] contains also an additional formula of this kind. As a consequence of this additional formula it is claimed that the modulus of monotonicity is a convex function. In the first part of this paper we give a simple example of a two-dimensional Banach lattice with a non-convex modulus of monotonicity. This disproves the claim made in [5] and shows that the modification of the Lindenstrauss formula given in [5] is false. We show that although the modulus of monotonicity need not be convex, it is continuous in the interval $[0, 1)$.

In the second part of this paper we study the non-strict Opial property and Opial property in general direct sums of Banach spaces. The Opial property was introduced in [8] and has many applications in metric fixed point theory (see [3]). Constructing a general direct sum, we use a function lattice called a substitution space. In the most standard cases this is the space $\mathbb{R}^n$ with a lattice norm or l_p space. We establish a result on permanence of Opial properties under passing to a direct sum. In case of the Opial property it is necessary to assume that a substitution space E is uniformly monotone and in the proof its modulus of monotonicity is used. In [2], results on the non-strict Opial property and Opial property in direct sums were proved, but only for particular cases of substitution spaces. Our theorem generalizes these results.

2. Preliminaries. In this paper we consider only real Banach spaces. Given such space X, by B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. To describe general construction of a direct sums of Banach spaces we introduce some preliminary notation. Given a nonempty set of indices I, consider the space $\mathrm{Map}(I, \mathbb{R})$ of all functions from I to $\mathbb{R}$ with the standard operations and order. For a set $A \subset I$, by 1_A we denote the characteristic function of A.

Let $(E, \|\cdot\|_E)$ be a real Banach space such that E is a linear subspace of $\mathrm{Map}(I, \mathbb{R})$, which satisfies the following monotonicity assumption. If $f \in E$, $g \in \mathrm{Map}(I, \mathbb{R})$ and $|g| \le |f|$, then $g \in E$ and $\|g\|_E \le \|f\|_E$. This condition implies in particular that if there is a function $f \in E$ such that $f(i_0) \neq 0$ for a given $i_0 \in I$, then $1_{\{i_0\}} \in E$. Consequently, we can and do assume that all functions $f \in \mathrm{Map}(I, \mathbb{R})$ with finite supports $\operatorname{supp} f$ belong to E. Following [4], we call such space E a substitution space. By E_0 we denote the closure of linear subspace spanned by all functions $1_{\{i\}}$, where $i \in I$.

Now, given a substitution space E on a set I and a family $\{X_i\}_{i \in I}$ of Banach spaces, we define the direct sum,

$$Y = \left(\sum_{i \in I} X_i \right)_E$$

as the space of all functions $x \in \prod_{i\in I} X_i$, where $x(i) \in X_i$ for every $i \in I$ such that $\lfloor x \rceil \in E$, where $\lfloor x \rceil(i) = \|x(i)\|$ for $i \in I$ (compare for example [1, p. 5]). We endow Y with the norm given by the formula

$$\|x\| = \|\lfloor x \rceil\|_E.$$

Observe that if $E = E_0$, then for every $x \in Y$ and every $\gamma > 0$ there exists a finite set $A \subset \operatorname{supp} x$ such that $\|x - 1_A x\| \le \gamma$.

Let us list some standard examples of substitution spaces, which give also standard particular cases of direct sums. If $I = \{1, \ldots, n\}$, we can take $E = \operatorname{Map}(I, \mathbb{R}) = \mathbb{R}^n$ endowed with a monotone norm. Then, trivially, $E_0 = E$.

Let now I be an infinite set. The standard and most frequently used substitution space is $E = l_p(I)$, where $1 \le p < \infty$. Also in this case $E_0 = E$. As the next example we take the space $E = l_\infty(I)$ of all bounded functions $f : I \to \mathbb{R}$ with the norm $\|f\| = \sup_{i\in I} |f(i)|$. In this case $E_0 = c_0(I)$ and this is another example of a substitution space.

The following simple remark will be used in the proof of our main result.

Remark 1. Consider a direct sum $Y = \left(\sum_{i\in I} X_i\right)_E$ and a finite set $I_0 \subset I$. Assume that for a sequence (x_n) in Y the following limits exist

$$\xi(i) = \lim_{n\to\infty} \|x_n(i)\|$$

for every $i \in I_0$. It is easy to see that

$$\lim_{n\to\infty} \big|\|x_n\| - \|1_{I_0}\xi + 1_{I\setminus I_0}\lfloor x_n \rceil\|\big| = 0.$$

It follows that if the limit $\lim_{n\to\infty} \|x_n\|$ exists, then

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|1_{I_0}\xi + 1_{I\setminus I_0}\lfloor x_n \rceil\|.$$

3. Modulus of monotonicity. Substitution spaces are Banach lattices, so dealing with a direct sum we use properties related to the order in a lattice E. Let us therefore recall some basic notation and terminology from the Banach lattice theory. More information on this subject can be found in the monographs [6] and [7].

Given a Banach lattice X, by X_+ we denote the non-negative cone, i.e. $X_+ = \{x \in X : x \ge 0\}$. Next, we put $B(X_+) = B_X \cap X_+$ and $S(X_+) = S_X \cap X_+$.

Recall that a Banach lattice X (or its norm) is strictly monotone if the conditions $0 \le x \le y$ and $x \ne y$ imply $\|x\| < \|y\|$. A strengthened version of strict monotonicity is called uniform monotonicity. It can be defined with the help of the following *modulus of monotonicity* of a Banach lattice X:

$$\delta_{m,X}(\epsilon) = \inf\{1 - \|x - y\| : 0 \le y \le x,\ \|x\| \le 1,\ \|y\| \ge \epsilon\}$$

where $\epsilon \in [0, 1]$. It is easy to establish the following basic properties of this modulus.

- We have $\delta_{m,X}(0) = 0 \le \delta_{m,X}(\epsilon) \le \epsilon$.
- In the definition of $\delta_{m,X}(\epsilon)$ the conditions $\|x\| \le 1$, $\|y\| \ge \epsilon$ can be replaced by $\|x\| = 1$, $\|y\| = \epsilon$.
- The function $\delta_{m,X}$ is non-decreasing.
- If $0 \le u \le v$ and $v \ne 0$, then

$$\delta_{m,X}\left(\frac{\|v-u\|}{\|v\|}\right) \le 1 - \left\|\frac{u}{\|v\|}\right\|$$

and consequently,

$$\|v\|\delta_{m,X}\left(\frac{\|v-u\|}{\|v\|}\right) \le \|v\| - \|u\|. \tag{3.1}$$

- A lattice X is strictly monotone if and only if $\delta_{m,X}(1) = 1$.

A lattice X is uniformly monotone, if $\delta_{m,X}(\epsilon) > 0$ for every $\epsilon > 0$. In case X is finite dimensional, uniform monotonicity is equivalent to strict monotonicity.

Another geometric property of Banach lattices is order uniform smoothness introduced in [5]. Given a Banach lattice X, we put

$$\rho_{m,X}(t) = \sup\{\|x \vee ty\| - 1 : x, y \in B(X_+)\},$$

where $t \ge 0$. A lattice X is order uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_{m,X}(t)}{t} = 0.$$

Order uniform smoothness is dual to uniform monotonicity. It is a consequence of the following counterpart of the Lindenstrauss formula given in Theorem 3(c) [5]:

$$\rho_{m,X^*}(t) = \sup\{\epsilon t - \delta_{m,X}(\epsilon) : 0 \le \epsilon \le 1\}. \tag{3.2}$$

From (3.2) it follows in particular that ρ_{m,X^*} is convex as a supremum of a family of convex functions. Theorem 3(d) [5] contains a formula obtained from (3.2) by interchanging ρ and δ. As a consequence, in Proposition 4 [5] it is claimed that $\delta_{m,X}$ is a convex functions. However, as we shall see below, this is not true. Consequently, also the formula from Theorem 3(d) [5] is false.

Example 1. To construct a lattice with a non-convex modulus of monotonicity we consider the space $X = \mathbb{R}^2$ with the norm

$$\|x\| = \max\left\{|x_1| + \frac{4}{9}|x_2|, \frac{3}{8}|x_1| + |x_2|\right\},$$

where $x = (x_1, x_2)$.

The unit sphere S_X is the octagon with vertices $(\pm 1, 0)$, $(0, \pm 1)$, $\left(\pm\frac{2}{3}, \pm\frac{3}{4}\right)$. The positive part $S(X_+)$ of the unit sphere consists of two segments: $s_1 = \left[(0,1), \left(\frac{2}{3}, \frac{3}{4}\right)\right]$ and $s_2 = \left[\left(\frac{2}{3}, \frac{3}{4}\right), (1,0)\right]$. It is therefore a graph of the function $\delta_1(x_1) = \min\{1 - \frac{3}{8}x_1, \frac{9}{4}(1 - x_1)\}$ (see Figure 1). Interchanging the axes,

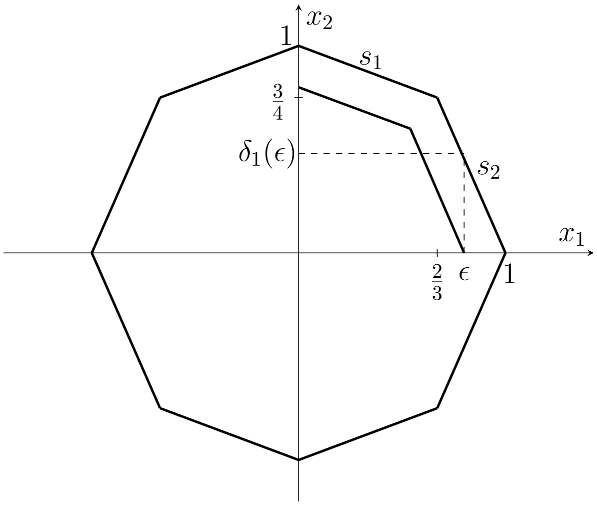

FIGURE 1

we can consider $S(X_+)$ as the graph of $\delta_2(x_2) = \min\{1 - \frac{4}{9}x_2, \frac{8}{3}(1 - x_2)\}$.

Clearly, $\delta_{m,X}(\epsilon) = 1 - \max \|x - y\|$, where the maximum is taken over all points $x = (x_1, x_2)$, $y = (y_1, y_2)$ such that $0 \le y \le x$, $\|y\| = \epsilon$, $\|x\| = 1$. It is easy to see that when y runs over ϵs_2, the maximal value of $\|x - y\|$ is attained at $y = (\epsilon, 0)$ and $x = (\epsilon, \delta_1(\epsilon))$. For such points $\|x - y\| = \delta_1(\epsilon)$. Analogously, when y runs over ϵs_1, the maximal value of $\|x - y\|$ equals $\delta_2(\epsilon)$. It follows that $\delta_{m,X}(\epsilon) = \min\{1 - \delta_1(\epsilon), 1 - \delta_2(\epsilon)\}$ which is not a convex function (see Figure 2).

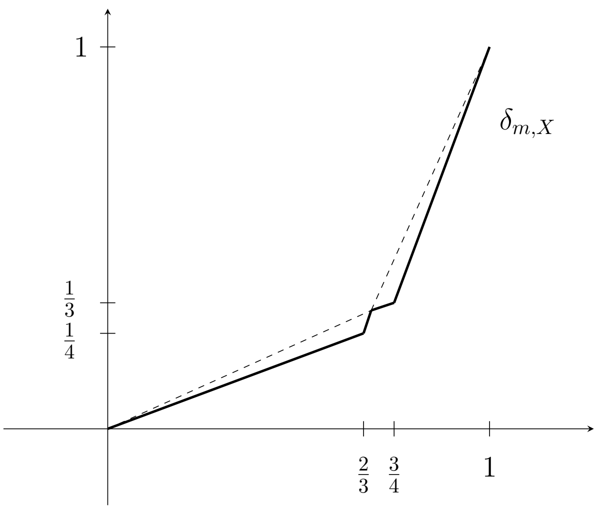

FIGURE 2

In the explicit form

$$\delta_{m,X}(\epsilon) = \begin{cases} \frac{3}{8}\epsilon & \text{if } \epsilon \in \left[0, \frac{2}{3}\right), \\ \frac{9}{4}\epsilon - \frac{5}{4} & \text{if } \epsilon \in \left[\frac{2}{3}, \frac{9}{13}\right), \\ \frac{4}{9}\epsilon & \text{if } \epsilon \in \left[\frac{9}{13}, \frac{3}{4}\right), \\ \frac{8}{3}\epsilon - \frac{5}{3} & \text{if } \epsilon \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Since, in general, $\delta_{m,X}$ need not be convex, the question about its continuity appears. The answer is positive: $\delta_{m,X}$ is continuous in the interval $[0,1)$. To see this it is enough to represent $\delta_{m,X}$ as an infimum of a family of convex functions. For this purpose, given $u, v \in S(X_+)$ and $\epsilon \in [0,1)$, we put $\delta_{u,v}(\epsilon) = 1 - \lambda$, where $\lambda \in [0,1]$ is the unique number for which $\|\epsilon u + \lambda v\| = 1$. Then $\delta_{u,v}$ is a convex function on $[0,1)$ and

$$\delta_{m,X}(\epsilon) = \inf\{\delta_{u,v}(\epsilon) : u, v \in S(X_+)\} \tag{3.3}$$

for every $\epsilon \in [0,1)$.

Now from (3.3) we can easily conclude that

$$|\delta_{m,X}(\epsilon_1) - \delta_{m,X}(\epsilon_2)| \leq \frac{1}{1-a}|\epsilon_1 - \epsilon_2|$$

for all $\epsilon_1, \epsilon_2 \in [0,a]$, $a < 1$, which in particular shows that $\delta_{m,X}$ is continuous in $[0,1)$.

4. Opial property in direct sums. Recall that a Banach space X has the *non-strict Opial property* if

$$\liminf_{n\to\infty} \|x_n\| \leq \liminf_{n\to\infty} \|x_n - x\|$$

for every weakly null sequence (x_n) in X and every $x \in X$.

If

$$\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n - x\|$$

for every weakly null sequence (x_n) in X and every non-zero $x \in X$, we say that X has the *Opial property*.

We show that these properties are preserved under passing to direct sums.

Theorem 1. *Let E be a substitution space on I such that $E_0 = E$, $\{X_i\}_{i\in I}$ be a family of Banach spaces and*

$$Y = \left(\sum_{i\in I} X_i\right)_E .$$

(i) *If all spaces X_i have the non-strict Opial property, then also Y has the non-strict Opial property.*
(ii) *If E is uniformly monotone and all spaces X_i have the Opial property, then also Y has the Opial property.*

Proof. Let (x_n) be a weakly null sequence in Y and $x \in Y$. Since $E = E_0$, for every $\gamma > 0$ there is a finite set $I_0 \subset \operatorname{supp} x$ such that

$$\|x - 1_{I_0} x\| \le \gamma.$$

Passing to a subsequence, we can assume that the following limits exist:

$$\lim_{n\to\infty} \|x_n\|, \quad \lim_{n\to\infty} \|x_n - x\|,$$

and

$$\xi(i) = \lim_{n\to\infty} \|x_n(i)\|, \quad \zeta(i) = \lim_{n\to\infty} \|x_n(i) - x(i)\|$$

for every $i \in I_0$. Assuming that all spaces X_i have the non-strict Opial property, we obtain the inequality $\xi(i) \le \zeta(i)$ for every $i \in I_0$. Consequently, the following inequality

$$u_n = 1_{I_0}\xi + 1_{I\setminus I_0}\lfloor x_n\rceil \le v_n = 1_{I_0}\zeta + 1_{I\setminus I_0}\lfloor x_n\rceil$$

holds in E. Using Remark 1, we therefore get

$$\begin{aligned}\lim_{n\to\infty} \|x_n\| &= \lim_{n\to\infty} \|u_n\|_E \le \limsup_{n\to\infty} \|v_n\|_E \\ &= \limsup_{n\to\infty} \|x_n - 1_{I_0} x\| \\ &\le \lim_{n\to\infty} \|x_n - x\| + \|x - 1_{I_0} x\| \\ &\le \lim_{n\to\infty} \|x_n - x\| + \gamma.\end{aligned}$$

Passing to the limit with $\gamma \to 0$, we obtain the conclusion of part (i).

To prove part (ii) we assume that E is uniformly monotone and all spaces X_i have the Opial property. Let (x_n) be a weakly null sequence in Y and $x \in Y \setminus \{0\}$. We put $M = \|x\| + 2\sup_{n\in\mathbb{N}} \|x_n\|$ and fix $i_0 \in \operatorname{supp} x$. Then $x(i_0) \ne 0$. For every $\gamma > 0$ there is a finite set $J_0 \subset \operatorname{supp} x$ such that $\|x - 1_{J_0} x\| \le \gamma$. Clearly, we also have $\|x - 1_{I_0} x\| \le \gamma$ where $I_0 = J_0 \cup \{i_0\}$.

In what follows we keep the notation and assumptions from the first part of the proof. Since $(x_n(i))$ is a weakly null sequence in X_i, the inequality $\zeta(i) \ge \|x(i)\|$ holds for every $i \in I_0$ and hence

$$\|v_n\|_E \ge \|1_{I_0}\zeta\|_E \ge \|1_{I_0} x\| \ge \|x(i_0)\| > 0.$$

On the other hand,

$$\|v_n\|_E \le \|1_{I_0}\zeta\|_E + \sup_{n\in\mathbb{N}} \|1_{I\setminus I_0} x_n\| \le M.$$

Using (3.1), we therefore obtain

$$\begin{aligned}\|u_n\|_E &\le \|v_n\|_E - \delta_{m,E}\left(\frac{\|v_n - u_n\|_E}{\|v_n\|_E}\right)\|v_n\|_E \\ &\le \|v_n\|_E - \delta_{m,E}\left(\frac{\|1_{I_0}(\zeta - \xi)\|_E}{M}\right)\|x(i_0)\| \le \|v_n\|_E - c\end{aligned}$$

where $c = \delta_{m,E}\left(\frac{\zeta(i_0)-\xi(i_0)}{M}\right)\|x(i_0)\|$. By our assumption X_{i_0} has the Opial property, so $\zeta(i_0) > \xi(i_0)$ and consequently, $c > 0$. Applying Remark 1, we conclude that

$$\begin{aligned}\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|u_n\|_E &\le \limsup_{n\to\infty}\|v_n\|_E - c \\ &\le \lim_{n\to\infty}\|x_n - x\| + \|x - 1_{I_0}x\| - c \\ &\le \lim_{n\to\infty}\|x_n - x\| + \gamma - c.\end{aligned}$$

Finally, passing to the limit with $\gamma \to 0$, we get

$$\lim_{n\to\infty}\|x_n\| \le \lim_{n\to\infty}\|x_n - x\| - c < \lim_{n\to\infty}\|x_n - x\|,$$

which gives us the conclusion of part (ii). □

Now we give examples showing that the assumptions imposed in Theorem 1 can not be dropped. In these examples we treat sequence spaces as direct sums of copies of the space $\mathbb{R}$, so in this case $Y = E$. Trivially, $\mathbb{R}$ with the absolute value norm has the Opial property.

As the first example we consider $Y = E = l_\infty$. Then, $E_0 = c_0 \neq E$, so our assumption on E is not satisfied. It is easy to see that l_∞ does not have the non-strict Opial property, i.e., the conclusion of Theorem 1 (i) does not hold.

Our second example is the space $Y = E = c_0$. In this case $E_0 = E$, but $\delta_{m,E}(\epsilon) = 0$ for every $\epsilon \in [0, 1]$, so the assumption from Theorem 1 (ii) is not satisfied. Clearly, c_0 does not have the Opial property, i.e., the conclusion of Theorem 1 (ii) does not hold.

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Joanna Markowicz
Institute of Mathematics
Maria Curie-Skłodowska University
20-031 Lublin, Poland

Department of Applied Mathematics
and Computer Science
University of Life Sciences in Lublin
20-612 Lublin, Poland
e-mail: joanna.markowicz@up.lublin.pl

Stanisław Prus
Institute of Mathematics
Maria Curie-Skłodowska University
20-031 Lublin, Poland
e-mail: stanislaw.prus@umcs.lublin.pl

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