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RABHA M. EL-ASHWAH, MOHAMED K. AOUF and S. M. EL-DEEB

Inclusion and neighborhood properties of certain subclasses of p-valent functions of complex order defined by convolution

Abstract. In this paper we introduce and investigate three new subclasses of p-valent analytic functions by using the linear operator $D^m_{\lambda,p}(f * g)(z)$. The various results obtained here for each of these function classes include coefficient bounds, distortion inequalities and associated inclusion relations for (n,θ)-neighborhoods of subclasses of analytic and multivalent functions with negative coefficients, which are defined by means of a non-homogenous differential equation.

1. Introduction. Let $A_p(n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_k z^k \quad (n > p;\ p, n \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : |z| < 1\}$. The Hadamard product (or convolution) of the functions $f(z)$ given by (1.1), and $g(z) \in A_p(n)$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (n > p;\ p, n \in \mathbb{N}) \tag{1.2}$$

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is defined by

$$(f*g)(z)=z^p+\sum_{k=n}^{\infty}a_kb_kz^k=(g*f)(z). \tag{1.3}$$

For functions $f,g\in A_p(n)$, we define the linear operator $D^m_{\lambda,p}:A_p(n)\to A_p(n)$ ($\lambda\geq 0$; $p,n\in\mathbb{N}$; $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$) by

$$D^0_{\lambda,p}(f*g)(z)=(f*g)(z), \tag{1.4}$$

$$D^1_{\lambda,p}(f*g)(z)=D_{\lambda,p}(f*g)(z)=(1-\lambda)(f*g)(z)+\frac{\lambda z}{p}(f*g)'(z) \tag{1.5}$$

and (in general)

$$\begin{aligned}D^m_{\lambda,p}(f*g)(z)&=D_{\lambda,p}(D^{m-1}_{\lambda,p}(f*g)(z))\\&=(1-\lambda)D^{m-1}_{\lambda,p}(f*g)(z)+\frac{\lambda z}{p}\left(D^{m-1}_{\lambda,p}(f*g)\right)'(z)\\&=z^p+\sum_{k=n}^{\infty}\left[\tfrac{p+\lambda(k-p)}{p}\right]^m a_kb_kz^k\end{aligned} \tag{1.6}$$

($\lambda\geq 0$; $p,n\in\mathbb{N}$; $m\in\mathbb{N}_0$; $z\in U$).

The operator $D^m_{\lambda,1}(f*g)(z)=D^m_{\lambda}(f*g)(z)$ was introduced by Aouf and Seoudy [6].

We note that

(i) for $\lambda=1$ and $b_k=1$ (or $g(z)=\frac{z^p}{1-z}$), $D^m_{\lambda,p}(f*g)(z)=D^m_pf(z)$, where the operator D^m_p is the p-valent Salagean operator introduced and studied by Aouf and Mostafa [5], Kamali and Orhan [11] and Orhan and Kiziltunc [13];

(ii) for $b_k=1$ (or $g(z)=\frac{z^p}{1-z}$), $D^m_{\lambda,p}(f*g)(z)=D^m_{\lambda,p}f(z)$, where the operator $D^m_{\lambda,p}$ was introduced and studied by El-Ashwah and Aouf [8].

For a function $f(z)\in A_p(n)$, we have

$$\left(D^m_{\lambda,p}(f*g)(z)\right)^{(q)}=\delta(p,q)z^{p-q}+\sum_{k=n}^{\infty}\delta(k,q)\left[\tfrac{p+\lambda(k-p)}{p}\right]^m a_kb_kz^{k-q}, \tag{1.7}$$

($\lambda\geq 0$; $p,n\in\mathbb{N}$; $q,m\in\mathbb{N}_0$; $p>q$; $z\in U$), where

$$\delta(p,q)=\begin{cases}1, & (q=0),\\ p(p-1)\dots(p-q+1), & (q\neq 0).\end{cases} \tag{1.8}$$

We denote by $T_p(n)$ the subclass of $A_p(n)$ consisting of functions of the form

$$f(z)=z^p-\sum_{k=n}^{\infty}a_kz^k \quad (n>p;\ a_k\geq 0;\ p,n\in\mathbb{N}). \tag{1.9}$$

For a given function $g(z) \in A_p(n)$ defined by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (b_k > 0;\ n > p;\ p, n \in \mathbb{N}), \tag{1.10}$$

we now introduce a new subclass $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ of the class $T_p(n)$ of p-valently analytic functions, which consists of functions $f(z) \in T_p(n)$ satisfying the inequality

$$\left| \frac{1}{b} \left\{ \frac{z(D_{\lambda,p}^m (f*g)(z))^{(q+1)} + \gamma z^2 (D_{\lambda,p}^m (f*g)(z))^{(q+2)}}{(1-\gamma)(D_{\lambda,p}^m (f*g)(z))^{(q)} + \gamma z (D_{\lambda,p}^m (f*g)(z))^{(q+1)}} - (p-q) \right\} \right| < \beta \tag{1.11}$$

($\lambda \geq 0$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$; $0 \leq \gamma \leq 1$; $p > q$; $0 < \beta \leq 1$; $b \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$; $z \in U$).

We note that

(1) $C_0^q(g(z); n, 0, p, \lambda, 1, b) = S_g(p, n, b, q)$
(Prajapat et al. [14]);

(2) $C_\gamma^q \left(z^p + \sum\limits_{k=n+p}^{\infty} \left(\frac{k+\mu}{p+\mu} \right)^r z^k; n+p, 0, p, \lambda, 1, b \right) = S_{n,q}^p(\mu, r, \gamma, b)$
($\mu \geq 0$ and $r \in \mathbb{N}_0$) (Srivastava et al. [18]);

(3) $C_0^q \left(z^p + \sum\limits_{k=n+p}^{\infty} \left[1 + \frac{\zeta(k-p)}{p+r} \right]^\eta z^k; n+p, 0, p, \lambda, 1, b \right) = H_{n,q}^{p,r}(b, \zeta, \eta)$
($\zeta, \eta, r \in \mathbb{R}$; $\zeta \geq 0$, $\eta \geq 0$, $r \geq 0$) (Mahzoon and Latha [12]);

(4) $C_\gamma^q \left(\frac{z^p}{1-z}; n+p, 0, p, \lambda, \beta, b \right) = S_{n,p}^q(\gamma, \beta, b)$
(Altintaş et al. [2]);

(5) $C_0^q \left(z^p + \sum\limits_{k=n+p}^{\infty} \binom{\mu+k-1}{k-p} z^k; n+p, 0, p, \lambda, 1, b \right) = H_{n,q}^p(\mu, b)$
($\mu \geq 0$) (Raina and Srivastava [15]);

(6) $C_\gamma^q \left(\frac{z^p}{1-z}; n+p, 0, p, \lambda, p-q-\alpha, 1 \right)$
$= C_\gamma^q \left(\frac{z^p}{1-z}; n+p, 0, p, \lambda, 1, p-q-\alpha \right) = T_n(p, q, \alpha, \gamma)$
($0 \leq \alpha < p-q$) (Altintaş [1]);

(7) $C_\gamma^q(g(z); n, 0, p, \lambda, \beta, b) = C_\gamma^q(g(z); n, p, \beta, b)$
(Srivastava and Orhan [17] and Aouf [4]);

(8) $C_0^0 \left(\frac{z^p}{1-z}; n, m, p, \lambda, \beta, b \right) = T_{n-p}(m, p, \lambda, b, \beta)$
(El-Ashwah and Aouf [8]).

Also, we note that

$$(1)\quad C_\gamma^q\left(z^p+\sum_{k=n}^{\infty}\left[\tfrac{p+\ell+\zeta(k-p)}{p+\ell}\right]^s z^k; n,0,p,\lambda,\beta,b\right)$$
$$=C_\gamma^q(\zeta,\ell,s;n,p,\beta,b)$$
$$=\Bigg\{f\in T_p(n):\left|\frac{1}{b}\left\{\frac{z(I_p^s(\zeta,\ell)f(z))^{(q+1)}+\gamma z^2(I_p^s(\zeta,\ell)f(z))^{(q+2)}}{(1-\gamma)(I_p^s(\zeta,\ell)f(z))^{(q)}+\gamma z(I_p^s(\zeta,\ell)f(z))^{(q+1)}}-(p-q)\right\}\right|<\beta,$$
$$p,n\in\mathbb{N};\ q,s\in\mathbb{N}_0;\ 0\le\gamma\le 1;\ p>q;\ 0<\beta\le 1;$$
$$\ell,\zeta\ge 0;\ b\in\mathbb{C}^*;\ z\in U\Bigg\},$$

where $I_p^s(\zeta,\ell)$ is an extended multiplier transformation (see Cătaş [7]), defined by

$$I_p^s(\zeta,\ell)f(z)=z^p-\sum_{k=n}^{\infty}\left[\tfrac{p+\ell+\zeta(k-p)}{p+\ell}\right]^s a_k z^k$$

$(\ell,\zeta\ge 0;\ p\in\mathbb{N}$ and $s\in\mathbb{N}_0)$;

$$(2)\quad C_\gamma^q\left(\tfrac{z^p}{1-z};n,m,p,\lambda,\beta,b\right)$$
$$=C_\gamma^q(n,m,p,\lambda,\beta,b)$$
$$=\Bigg\{f\in T_p(n):\left|\frac{1}{b}\left\{\frac{z(D_{\lambda,p}^m f(z))^{(q+1)}+\gamma z^2(D_{\lambda,p}^m f(z))^{(q+2)}}{(1-\gamma)(D_{\lambda,p}^m f(z))^{(q)}+\gamma z(D_{\lambda,p}^m f(z))^{(q+1)}}-(p-q)\right\}\right|<\beta,$$
$$p,n\in\mathbb{N};\ q,m\in\mathbb{N}_0;\ b\in\mathbb{C}^*;\ 0\le\gamma\le 1;\ p>q;$$
$$0<\beta\le 1;\lambda\ge 0\Bigg\}.$$

Also let $R_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ denote the subclass $T_p(n)$ consisting of functions $f(z)$ of the form (1.9) and the function $g(z)$ of the form (1.10) which satisfy the following inequality:

$$(1.12)\qquad \left|\frac{1}{b}\left\{(1-\gamma)\frac{(D_{\lambda,p}^m(f*g)(z))^{(q)}}{z^{p-q}}+\gamma\frac{(D_{\lambda,p}^m(f*g)(z))^{(q+1)}}{(p-q)z^{p-q-1}}-\delta(p,q)\right\}\right|<\beta$$

$(\lambda\ge 0;\ p,n\in\mathbb{N};\ q,m\in\mathbb{N}_0;\ 0\le\gamma\le 1;\ p>q;\ 0<\beta\le 1;\ b\in\mathbb{C}^*;\ z\in U)$.

In this paper we shall study some properties of the classes $C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ and $R_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ and derive several results for functions in the subclass $H_\gamma^q(g(z);n,m,p,\lambda,\beta,b,\alpha)$ of the function class $T_p(n)$, which is defined as follows:

A function $f(z)\in T_p(n)$ is said to belong to the class $H_\gamma^q(g(z);n,m,p,\lambda,\beta,b,\alpha)$ if $w=f(z)$ satisfies the following non-homogenous Cauchy–Euler differential equation:

$$z^2\frac{d^{q+2}w}{dz^{q+2}}+2(1+\alpha)z\frac{d^{q+1}w}{dz^{q+1}}+\alpha(1+\alpha)\frac{d^qw}{dz^q} = (p-q+\alpha)(p-q+\alpha+1)\frac{d^qk}{dz^q}, \tag{1.13}$$

where $k(z)\in C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ and $\alpha>q-p$, $\alpha\in R$, $p\in\mathbb{N}$, $q\in\mathbb{N}_0$.

2. Basic properties of the classes $C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ and $R_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$. We begin by proving a necessary and sufficient condition for a function belonging to the class $T_p(n)$ to be in the class $C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$.

Theorem 1. *Let the function $f(z)\in T_p(n)$ be defined by* (1.9) *and let $g(z)$ be defined by* (1.10). *Then $f(z)$ is in the class $C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ if and only if*

$$\sum_{k=n}^{\infty}[k-p+\beta\,|b|]\,[1+\gamma(k-q-1)]\left[\tfrac{p+\lambda(k-p)}{p}\right]^m\delta(k,q)a_kb_k \le \beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q). \tag{2.1}$$

Proof. If the condition (2.1) holds true, we find from (1.9), (1.10) and (2.1) that

$$\begin{aligned}
&\Big|z(D_{\lambda,p}^m(f*g)(z))^{(q+1)}+\gamma z^2(D_{\lambda,p}^m(f*g)(z))^{(q+2)}\\
&\quad-(p-q)\Big[(1-\gamma)(D_{\lambda,p}^m(f*g)(z))^{(q)}+\gamma z(D_{\lambda,p}^m(f*g)(z))^{(q+1)}\Big]\Big|\\
&\quad-\beta\Big|b\Big[(1-\gamma)(D_{\lambda,p}^m(f*g)(z))^{(q)}+\gamma z(D_{\lambda,p}^m(f*g)(z))^{(q+1)}\Big]\Big|\\
&=\Big|\delta(p,q+1)z^{p-q}-\sum_{k=n}^{\infty}\left[\tfrac{p+\lambda(k-p)}{p}\right]^m\delta(k,q+1)a_kb_kz^{k-q}\\
&\quad+\gamma\delta(p,q+2)z^{p-q}-\sum_{k=n}^{\infty}\gamma\left[\tfrac{p+\lambda(k-p)}{p}\right]^m\delta(k,q+2)a_kb_kz^{k-q}\\
&\quad-(p-q)\Big[(1-\gamma)\delta(p,q)z^{p-q}-\sum_{k=n}^{\infty}(1-\gamma)\left[\tfrac{p+\lambda(k-p)}{p}\right]^m\delta(k,q)a_kb_kz^{k-q}\\
&\quad+\gamma\delta(p,q+1)z^{p-q}-\sum_{k=n}^{\infty}\gamma\left[\tfrac{p+\lambda(k-p)}{p}\right]^m\delta(k,q+1)a_kb_kz^{k-q}\Big]\Big|\\
&\quad-\beta\Big|b\Big[(1-\gamma)\delta(p,q)z^{p-q}-\sum_{k=n}^{\infty}(1-\gamma)\left[\tfrac{p+\lambda(k-p)}{p}\right]^m\delta(k,q)a_kb_kz^{k-q}\\
&\quad+\gamma\delta(p,q+1)z^{p-q}-\sum_{k=n}^{\infty}\gamma\left[\tfrac{p+\lambda(k-p)}{p}\right]^m\delta(k,q+1)a_kb_kz^{k-q}\Big]\Big|
\end{aligned}$$

$$= \left| \sum_{k=n}^{\infty} (k-p)\left[1+\gamma(k-q-1)\right] \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k z^{k-q} \right|$$
$$- \beta \left| b \left[(1+\gamma(p-q-1))\delta(p,q) z^{p-q} - \sum_{k=n}^{\infty} (1+\gamma(k-q-1)) \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k z^{k-q} \right] \right|$$
$$\leq \sum_{k=n}^{\infty} (k-p)\left[1+\gamma(k-q-1)\right] \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k \left|z\right|^{k-p}$$
$$- \beta \left|b\right| \left\{ \left[1+\gamma(p-q-1)\right]\delta(p,q) - \sum_{k=n}^{\infty} \left[1+\gamma(k-q-1)\right] \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k \left|z\right|^{k-p} \right\}$$
$$\leq \sum_{k=n}^{\infty} \left[k-p+\beta\left|b\right|\right] \left[1+\gamma(k-q-1)\right] \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k$$
$$- \beta \left|b\right| \left[1+\gamma(p-q-1)\right]\delta(p,q) \leq 0$$

$(z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\})$. Hence, by the maximum modulus theorem, $f(z) \in C_\gamma^q(g(z); n, p, \beta, b)$.

Conversely, let $f(z) \in C_\gamma^q(g(z); n, p, \beta, b)$ be given by (1.9) and $g(z)$ be given by (1.10). Then from (1.7) and (1.11), we have

$$\left| \frac{1}{b} \left\{ \frac{z(D_{\lambda,p}^m(f*g)(z))^{(q+1)} + \gamma z^2 (D_{\lambda,p}^m(f*g)(z))^{(q+2)}}{(1-\gamma)(D_{\lambda,p}^m(f*g)(z))^{(q)} + \gamma z (D_{\lambda,p}^m(f*g)(z))^{(q+1)}} - (p-q) \right\} \right|$$

$$(2.2) \qquad = \left| \frac{1}{b} \left\{ \frac{\sum_{k=n}^{\infty} (k-p)[1+\gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k z^{k-p}}{[1+\gamma(p-q-1)]\delta(p,q) - \sum_{k=n}^{\infty} [1+\gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k z^{k-p}} \right\} \right|$$
$$< \beta.$$

Putting $z = r$ $(0 \leq r < 1)$ on the right-hand side of (2.2) and noting the fact that for $r = 0$, the resulting expression in the denominator is positive and remains so for all $r \in (0, 1)$, the desired inequality (2.1) follows upon letting $r \to 1^-$. □

Theorem 2. *Let the function $f(z) \in T_p(n)$ be defined by* (1.9) *and $g(z)$ be defined by* (1.10). *Then $f(z)$ is in the class $R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ if and only if*

$$(2.3) \qquad \sum_{k=n}^{\infty} \left[p-q+\gamma(k-p)\right] \left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k \leq \beta \left|b\right| (p-q).$$

Corollary 1. *Let the function $f(z) \in T_p(n)$ be given by* (1.9) *and $g(z)$ be defined by* (1.10)*. If $f(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, then*

$$a_k \leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{[k-p+\beta\,|b|]\,[1+\gamma(k-q-1)]\left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) b_k} \tag{2.4}$$

($k \geq n$; $\lambda \geq 0$; $0 \leq \gamma \leq 1$; $0 < \beta \leq 1$; $b \in \mathbb{C}^*$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$).

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{\delta(k,q)\,[k-p+\beta\,|b|]\,[1+\gamma(k-q-1)]\left[\frac{p+\lambda(k-p)}{p}\right]^m b_k} z^k \tag{2.5}$$

($k \geq n$; $\lambda \geq 0$; $0 \leq \gamma \leq 1$; $0 < \beta \leq 1$; $b \in \mathbb{C}^*$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$).

We next prove the following growth and distortion property for the functions of the form (1.9) belonging to the class $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$.

Theorem 3. *If a function $f(z)$ defined by* (1.9) *is in the class $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and $g(z)$ defined by* (1.10)*. Then*

$$\begin{aligned} &||f(z)| - |z|^p| \\ &\leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m \delta(n,q) b_n} |z|^n \end{aligned} \tag{2.6}$$

($\lambda \geq 0$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$; $0 \leq \gamma \leq 1$; $n > p > q$; $0 < \beta \leq 1$; $b \in \mathbb{C}^*$; $z \in U$) and (in general)

$$\begin{aligned} &\left|\left|f^{(r)}(z)\right| - \delta(p,r)\,|z|^{p-r}\right| \\ &\leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,(n-q)!\delta(p,q)}{(n-p+\beta\,|b|)(n-r)!\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} |z|^{n-r} \end{aligned} \tag{2.7}$$

($z \in U$; $p, n \in \mathbb{N}$; $n > p$; $m, q \in \mathbb{N}_0$; $r \leq q < p$; $p > \max(r,q)$; $\lambda \geq 0$). The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m \delta(n,q) b_n} z^n \tag{2.8}$$

($n > p$; $p, n \in \mathbb{N}$).

Proof. In view of Theorem 1, we have

$$\begin{aligned} &(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\tfrac{p+\lambda(n-p)}{p}\right]^m \delta(n,q) b_n \sum_{k=n}^{\infty} a_k \\ &\leq \sum_{k=n}^{\infty} [k-p+\beta\,|b|]\,[1+\gamma(k-q-1)]\left[\tfrac{p+\lambda(k-p)}{p}\right]^m \delta(k,q) a_k b_k \\ &\leq \beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q), \end{aligned}$$

which readily yields

$$\sum_{k=n}^{\infty} a_k \leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m \delta(n,q)b_n}. \tag{2.9}$$

Also, (2.1) yields

$$\sum_{k=n}^{\infty} k!a_k \leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,(n-q)!\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n}. \tag{2.10}$$

Now, by differentiating r times both sides of (1.9), we have

$$f^{(r)}(z) = \delta(p,r)z^{p-r} - \sum_{k=n}^{\infty} \delta(k,r)a_k z^{k-r} \tag{2.11}$$

$(p, n \in \mathbb{N};\ r \in \mathbb{N}_0;\ p > r)$.

Theorem 3 follows from (2.9), (2.10) and (2.11). Finally, it is easy to see that the bounds in Theorem 1 are attained for the function $f(z)$ given by (2.8). □

3. Properties of the class $H^q_\gamma(g(z); n, m, p, \lambda, \beta, b, \alpha)$. Applying the results of Section 2, which are obtained for the function $f(z)$ of the form (1.9) belonging to the class $C^q_\gamma(g(z); n, m, p, \lambda, \beta, b)$, we now derive the corresponding results for the function $f(z)$ belonging to the class $H^q_\gamma(g(z); n, m, p, \lambda, \beta, b, \alpha)$.

Theorem 4. *If a function $f(z)$ is defined by* (1.9) *and $g(z)$ is defined by* (1.10)*, and $f(z)$ is in the class $H^q_\gamma(g(z); n, m, p, \lambda, \beta, b, \alpha)$. Then*

$$\begin{aligned} &\left||f(z)| - |z|^p\right| \\ &\leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,(p-q+\alpha)(p-q+\alpha+1)\delta(p,q)}{(n-p+\beta\,|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m (n-q+\alpha)\delta(n,q)b_n}\,|z|^n \end{aligned} \tag{3.1}$$

and (in general)

$$\begin{aligned} &\left|\left|f^{(r)}(z)\right| - \delta(p,r)\,|z|^{p-r}\right| \\ &\leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,(p-q+\alpha)(p-q+\alpha+1)(n-q)!\delta(p,q)}{(n-p+\beta\,|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m (n-q+\alpha)(n-r)!b_n}\,|z|^{n-r} \end{aligned} \tag{3.2}$$

$(p, n \in \mathbb{N};\ m, q \in \mathbb{N}_0;\ r \leq q < p;\ p > \max(r,q);\ 0 \leq \gamma \leq 1;\ 0 < \beta \leq 1;$ $b \in \mathbb{C}^*;\ \lambda \geq 0;\ z \in U)$. The results in (3.1) *and* (3.2) *are sharp for the function $f(z)$ given by*

$$f(z) = z^p - \frac{\beta\,|b|\,\delta(p,q)\,[1+\gamma(p-q-1)]\,(p-q+\alpha)(p-q+\alpha+1)}{(n+\beta|b|)\delta(n+p,q)[1+\gamma(n+p-q-1)](n+p-q+\alpha)b_{n+p}}z^n. \tag{3.3}$$

Proof. Assume that $f(z) \in T_p(n)$ is given by (1.9) *and* $g(z)$ given by (1.10). Also, let function $k(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, occurring in the non-homogenous differential equation (1.13) be of the form:

$$k(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \tag{3.4}$$

$(c_k \geq 0;\ n > p;\ p, n \in \mathbb{N})$. Then, we readily find from (1.13) that

$$a_k = \frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)} c_k \tag{3.5}$$

$(k \geq n; p, n \in \mathbb{N})$, so that

$$f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k = z^p - \sum_{k=n}^{\infty} \frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)} c_k z^k \tag{3.6}$$

$(z \in U)$, and

$$||f(z)| - |z|^p| \leq |z|^n \sum_{k=n}^{\infty} \frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)} c_k \tag{3.7}$$

$(z \in U)$. Next, since $k(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, therefore, on using the assertion (2.4) of Corollary 1, we get the following coefficient inequality:

$$c_k \leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m \delta(n,q) b_n} \tag{3.8}$$

$(k \geq n;\ n > p > q;\ \lambda \geq 0;\ 0 \leq \gamma \leq 1;\ 0 < \beta \leq 1;\ p, n \in \mathbb{N};\ q, m \in \mathbb{N}_0;$ $b \in \mathbb{C}^*)$, which in conjunction with (3.6) and (3.7) yields

$$\begin{aligned} &||f(z)| - |z|^p| \\ &\quad \leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,(p-q+\alpha)(p-q+\alpha+1)\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m \delta(n,q) b_n} |z|^n \\ &\quad \times \sum_{k=n}^{\infty} \frac{1}{(k-q+\alpha)(k-q+\alpha+1)} \end{aligned} \tag{3.9}$$

$(z \in U)$. Note that the following summation result holds

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{1}{(k-q+\alpha)(k-q+\alpha+1)} &= \sum_{k=n}^{\infty} \left(\frac{1}{(k-q+\alpha)} - \frac{1}{(k-q+\alpha+1)} \right) \\ &= \frac{1}{(n-q+\alpha)}, \end{aligned} \tag{3.10}$$

where $\alpha \in \mathbb{R}^* = \mathbb{R} \backslash \{-n, -n-1, \ldots\}$. The assertion (3.1) of Theorem 4 follows from (3.9) and (3.10), respectively. The assertion (3.2) of Theorem 4 can be established similarly by applying (2.10), (2.11), (3.5) and (3.10), respectively. □

4. Inclusion relations involving (n,θ)-neighborhood for the classes $C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$, $R_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ and $H_\gamma^q(g(z);n,m,p,\lambda,\beta,b,\alpha)$. Following the works of Goodman [10], Ruscheweyh [16] and Altintaş [1] (see also [2], [3] and [9]), we define the (n,θ)-neighborhood of a function $f^{(q)}(z)$ when $f\in T_p(n)$ by

$$N_{n,p}^{\theta}(f^{(q)},k^{(q)}) = \left\{k\in T_p(n): k(z)=z^p-\sum_{k=n}^{\infty}c_kz^k \text{ and } \sum_{k=n}^{\infty}k\delta(k,q)\left|a_k-c_k\right|\le\theta\right\}. \tag{4.1}$$

It follows from (4.1) that, if

$$h(z)=z^p \tag{4.2}$$

$(p\in\mathbb{N})$, then

$$N_{n,p}^{\theta}(h^{(q)}) = \left\{k\in T_p(n): k(z)=z^p-\sum_{k=n}^{\infty}c_kz^k \text{ and } \sum_{k=n+p}^{\infty}k\delta(k,q)\left|c_k\right|\le\theta\right\}. \tag{4.3}$$

Next, we establish inclusion relationships for the function classes $C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ and $R_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$, involving the (n,θ)-neighborhood $N_{n,p}^{\theta}(h^{(q)})$ defined by (4.3).

Theorem 5. *If $b_k\ge b_n$ $(k\ge n)$ and*

$$\theta=\frac{n\beta\left|b\right|\left[1+\gamma(p-q-1)\right]\delta(p,q)}{(n-p+\beta\left|b\right|)\left[1+\gamma(n-q-1)\right]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} \tag{4.4}$$

$(p>|b|)$, then

$$C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)\subset N_{n,p}^{\theta}(h^{(q)}). \tag{4.5}$$

Proof. Let $f\in C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$. Then, in view of the assertion (2.1) of Theorem 1, and the given condition that $b_k\ge b_n$ $(k\ge n)$, we have

$$\begin{aligned}(n-p+\beta\left|b\right|)\left[1+\gamma(n-q-1)\right]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n\sum_{k=n}^{\infty}\delta(k,q)a_k \\ \le\beta\left|b\right|\left[1+\gamma(p-q-1)\right]\delta(p,q)\end{aligned} \tag{4.6}$$

so that

$$\sum_{k=n}^{\infty}\delta(k,q)a_k\le\frac{\beta\left|b\right|\left[1+\gamma(p-q-1)\right]\delta(p,q)}{(n-p+\beta\left|b\right|)\left[1+\gamma(n-q-1)\right]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n}. \tag{4.7}$$

On the other hand, we also find from (2.1) and (4.7) that

$$\begin{aligned}\sum_{k=n}^{\infty} k\delta(k,q)a_k &\leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} + (p-\beta\,|b|)\sum_{k=n}^{\infty}\delta(k,q)a_k \\ &\leq \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} \\ &\quad + \frac{(p-\beta\,|b|)\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n},\end{aligned}$$

that is

$$\sum_{k=n}^{\infty}\delta(k,q)ka_k \leq \frac{n\beta\,|b|\,\delta(p,q)\,[1+\gamma(p-q-1)]}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} = \theta. \tag{4.8}$$

This evidently completes the proof of Theorem 5. □

Remark 1. (i) Taking $g(z)=\frac{z^p}{1-z}$, $b=\gamma$, $m=0$ and $\gamma=\lambda$ in Theorem 5, we obtain the result obtained by Altintaş et al. [2, Theorem 2];

(ii) Taking $g(z)=\frac{z^p}{1-z}$, $b=1$, $\beta=p-\alpha$ $(0\leq\alpha<p)$ and $\gamma=\lambda$ in Theorem 5, we obtain the result obtained by Altintaş [1, Theorem 2].

Putting $g(z)=z^p+\sum_{k=n}^{\infty}\left[\frac{p+\ell+\zeta(k-p)}{p+\ell}\right]^s z^k$ $(\ell,\zeta\geq 0;\ s\in\mathbb{N}_0)$ and $m=0$ in Theorem 5, we obtain the following corollary.

Corollary 2. *If $f(z)\in T_p(n)$ is in the class $C_\gamma^q(\zeta,\ell,s;n,p,\beta,b)$, then*

$$C_\gamma^q(\zeta,\ell,s;n,p,\beta,b)\subset N_{n,p}^{\theta}(h^{(q)}),$$

where $h(z)$ is given by (4.2) and

$$\theta=\frac{n\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]}\left(\frac{p+\ell}{p+\ell+\zeta(n-p)}\right)^s.$$

Putting $g(z)=z^p+\sum_{k=n}^{\infty}\left[\frac{p+\zeta(k-p)}{p}\right]^s z^k$ $(\zeta\geq 0;\ s\in\mathbb{N}_0)$ and $m=0$ in Theorem 5, we obtain the following corollary.

Corollary 3. *If $f(z)\in T_p(n)$ is in the class $C_\gamma^q(\zeta,s;n,p,\beta,b)$, then*

$$C_\gamma^q(\zeta,s;n,p,\beta,b)\subset N_{n,p}^{\theta}(h^{(q)}),$$

where $h(z)$ is given by (4.2) and

$$\theta=\frac{n\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]}\left(\frac{p}{p+\zeta(n-p)}\right)^s.$$

Theorem 6. *If*

$$\theta = \frac{n\beta\,|b|\,(p-q)}{[p-q+\gamma(n-p)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n}, \tag{4.9}$$

then

$$R_\gamma^q(g(z);n,m,p,\lambda,\beta,b) \subset N_{n,p}^\theta\big(h^{(q)}\big). \tag{4.10}$$

Proof. Let $f \in R_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$. Then, in view of the assertion (2.3) of Theorem 2, we have

$$\begin{aligned} &\frac{[p-q+\gamma(n-p)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n}{n}\sum_{k=n}^{\infty}\delta(k,q)ka_k \\ &\leq \sum_{k=n}^{\infty}[p-q+\gamma(k-p)]\left[\frac{p+\lambda(k-p)}{p}\right]^m \delta(k,q)a_kb_k \\ &\leq \beta\,|b|\,(p-q), \end{aligned}$$

so that

$$\sum_{k=n}^{\infty}\delta(k,q)ka_k \leq \frac{n\beta\,|b|\,(p-q)}{[p-q+\gamma(n-p)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} = \theta, \tag{4.11}$$

which by means of the definition (4.1), establishes the inclusion (4.10) asserted by Theorem 6. □

Theorem 7. *If $f(z) \in T_p(n)$ is in the class $H_\gamma^q(g(z);n,m,p,\lambda,\beta,b,\alpha)$, then*

$$H_\gamma^q(g(z);n,m,p,\lambda,\beta,b,\alpha) \subset N_{n,p}^\theta(f^{(q)},k^{(q)}), \tag{4.12}$$

where $k(z)$ is given by (1.13) *and*

$$\theta = \frac{n\beta\,|b|\,[1+\gamma(p-q-1)]\,[n+(p-q+\alpha)(p-q+\alpha+2)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m(n-q+\alpha)b_n}. \tag{4.13}$$

Proof. Suppose that $f(z) \in H_\gamma^q(g(z);n,m,p,\lambda,\beta,b,\alpha)$. Then upon substituting from (3.5) into the following coefficient inequality

$$\sum_{k=n}^{\infty}k\delta(k,q)\,|a_k-c_k| \leq \sum_{k=n}^{\infty}k\delta(k,q)\,|c_k| + \sum_{k=n}^{\infty}k\delta(k,q)\,|a_k| \tag{4.14}$$

$(a_k;\ c_k \geq 0)$, we readily obtain

$$\begin{aligned}(4.15)\qquad \sum_{k=n}^{\infty} k\delta(k,q)\,|a_k - c_k| \leq & \sum_{k=n}^{\infty} k\delta(k,q)\,|c_k| \\ & + \sum_{k=n}^{\infty} k\delta(k,q)\frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)}\,|c_k|\,.\end{aligned}$$

Now, since $k(z) \in C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ the second assertion (4.8) yields

$$(4.16)\qquad k\delta(k,q)c_k \leq \frac{n\beta\,|b|\,\delta(p,q)\,[1+\gamma(p-q-1)]}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n}.$$

Finally, by making use of (4.8) as well as (4.16) on the right-hand side of (4.15), we find that

$$\begin{aligned}&\sum_{k=n+p}^{\infty} \delta(k,q)k\,|a_k - c_k| \\ &\leq \frac{n\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} \\ &\quad\times\left(1+\sum_{k=n+p}^{\infty}\frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)}\right) \\ &= \frac{n\beta\,|b|\,[1+\gamma(p-q-1)]\,[n+(p-q+\alpha)(p-q+\alpha+2)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\,(n+p-q+\alpha)\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n} \\ &= \theta,\end{aligned}$$

we conclude that $f \in N_{n,p}^{\theta}(f^{(q)}, k^{(q)})$. This evidently completes the proof of Theorem 7. □

5. Neighborhood for the classes $C_\gamma^{q,\zeta}(g(z);n,m,p,\lambda,\beta,b)$ and $R_\gamma^{q,\zeta}$ $(g(z);n,m,p,\lambda,\beta,b)$. In this section we determine the neighborhood for the classes $C_\gamma^{q,\zeta}(g(z);n,m,p,\lambda,\beta,b)$ and $R_\gamma^{q,\zeta}(g(z);n,m,p,\lambda,\beta,b)$ which we define as follows. A function $f \in T_p(n)$ is said to be in the class $C_\gamma^{q,\zeta}(g(z);n,m,p,\lambda,\beta,b)$ if there exists a function $k \in C_\gamma^q(g(z);n,m,p,\lambda,\beta,b)$ such that

$$(5.1)\qquad \left|\frac{f(z)}{k(z)} - 1\right| < p - \zeta$$

$(z \in U;\ 0 \leq \zeta < p)$.

Theorem 8. *If* $k(z) \in C_{\gamma}^{q}(g(z); n, m, p, \lambda, \beta, b)$ *and*

$$\zeta = p - \frac{\theta(n-p+\beta|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^{m} b_n}{n\left\{(n-p+\beta|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^{m}\delta(n,q)b_n - \beta|b|[1+\gamma(p-q-1)]\delta(p,q)\right\}},$$

then

$$N_{n,p}^{\theta}\big(k^{(q)}\big) \subset C_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b), \tag{5.2}$$

where

$$\theta \le np\Bigg[\delta(n,q) - \beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q) \times \left\{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^{m} b_n\right\}^{-1}\Bigg].$$

Proof. Suppose that $f \in N_{n,p}^{\theta}\big(k^{(q)}\big)$, then we find from the definition (4.1) that

$$\sum_{k=n}^{\infty} \delta(k,q)k\,|a_k - c_k| \le \theta, \tag{5.3}$$

which implies the coefficient inequality

$$\sum_{k=n}^{\infty} |a_k - c_k| \le \frac{\theta}{n\delta(n,q)} \tag{5.4}$$

$(p > q;\ n, p \in \mathbb{N},\ q \in \mathbb{N}_0)$. Next, since $k(z) \in C_{\gamma}^{q}(g(z); n, m, p, \lambda, \beta, b)$, we have

$$\sum_{k=n}^{\infty} c_k \le \frac{\beta\,|b|\,[1+\gamma(p-q-1)]\,\delta(p,q)}{(n-p+\beta\,|b|)\,[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^{m}\delta(n,q)b_n},$$

so that

$$\begin{aligned}
\left|\frac{f(z)}{k(z)} - 1\right| &\le \frac{\sum\limits_{k=n}^{\infty}|a_k - c_k|}{1 - \sum\limits_{k=n}^{\infty}|c_k|} \\
&\le \frac{\frac{\theta}{n\delta(n,q)}}{1 - \frac{\beta|b|[1+\gamma(p-q-1)]\delta(p,q)}{(n-p+\beta|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^{m}\delta(n,q)b_n}} \\
&= \frac{\theta(n-p+\beta|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^{m} b_n}{n\left\{(n-p+\beta|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^{m}\delta(n,q)b_n - \beta|b|\delta(p,q)[1+\gamma(p-q-1)]\right\}} \\
&= p - \zeta,
\end{aligned}$$

because by the assumption

$$\zeta = p - \frac{\theta(n-p+\beta|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n}{n\left\{(n-p+\beta|b|)[1+\gamma(n-q-1)]\left[\frac{p+\lambda(n-p)}{p}\right]^m \delta(n,q)b_n - \beta|b|\delta(p,q)[1+\gamma(p-q-1)]\right\}}.$$

This implies that $f \in C_\gamma^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b)$. □

Similarly, we can prove the following theorem.

Theorem 9. *If* $k(z) \in R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ *and*

$$\zeta = p - \frac{\theta\left[p-q+\gamma(n-p)\right]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n}{n\left\{\left[p-q+\gamma(n-p)\right]\left[\frac{p+\lambda(n-p)}{p}\right]^m \delta(n,q)b_n - \beta\left|b\right|(p-q)\right\}}, \tag{5.5}$$

then

$$N_{n,p}^{\theta}(k^{(q)}) \subset R_\gamma^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b), \tag{5.6}$$

where

$$\theta \le np\left[\delta(n,q) - \beta\left|b\right|(p-q)\left\{\left[p-q+\gamma(n-p)\right]\left[\frac{p+\lambda(n-p)}{p}\right]^m b_n\right\}^{-1}\right].$$

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R. M. El-Ashwah
Department of Mathematics
Faculty of Science at Damietta
University of Mansoura
New Damietta 34517
Egypt
e-mail: r_elashwah@yahoo.com

M. K. Aouf
Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura 33516
Egypt
e-mail: mkaouf127@yahoo.com

S. M. El-Deeb
Department of Mathematics
Faculty of Science at Damietta
University of Mansoura
New Damietta 34517
Egypt
e-mail: shezaeldeeb@yahoo.com

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