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Multiplication formulas for q-Appell polynomials and the multiple q-power sums

Abstract. In the first article on q-analogues of two Appell polynomials, the generalized Apostol-Bernoulli and Apostol-Euler polynomials, focus was on generalizations, symmetries, and complementary argument theorems. In this second article, we focus on a recent paper by Luo, and one paper on power sums by Wang and Wang. Most of the proofs are made by using generating functions, and the (multiple) q-addition plays a fundamental role. The introduction of the q-rational numbers in formulas with q-additions enables natural q-extension of vector forms of Raabes multiplication formulas. As special cases, new formulas for q-Bernoulli and q-Euler polynomials are obtained.

1. Introduction. In 2006, Luo and Srivastava [8, p. 635-636] found new relationships between Apostol–Bernoulli and Apostol–Euler polynomials. This was followed by the pioneering article by Luo [10], where multiplication formulas for the Apostol–Bernoulli and Apostol–Euler polynomials of higher order, together with λ-multiple power sums were introduced. Luo also expressed these λ-multiple power sums as sums of the above polynomials. One year later, Wang and Wang [12] introduced generating functions for λ-power sums, some of the proofs use a symmetry reasoning, which lead

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to many four-line identities for Apostol–Bernoulli and Apostol–Euler polynomials and λ-power sums; as special cases, some of the above Luo identities were obtained.

In [5] it was proved that the q-Appell polynomials form a commutative ring; in this paper we show what this means in practice. Thus, the aim of the present paper is to find q-analogues of most of the above formulas with the aid of the multiple q-addition, the q-rational numbers, and so on. Many formulas bear a certain resemblance to the q-Taylor formula, where q-rational numbers appear to the right in the function argument; this means that the alphabet is extended to $\mathbb{Q}_{\oplus_q}$. In some proofs, both q-binomial coefficients and a vector binomial coefficient occur, this is connected to a vector form of the multinomial theorem, with binomial coefficients, unlike the case in [3, p. 110].

This paper is organized as follows: In this section we give the general definitions. In each section, we then give the specific definitions and special values which we use there.

In Section 2, multiple q-Apostol–Bernoulli polynomials and q-power sums are introduced and multiplication formulas for q-Apostol–Bernoulli polynomials are proved, which are q-analogues of Luo [10].

In Section 3, multiplication formulas for q-Apostol–Euler polynomials are proved. In Section 4, formulas containing q-power sums in one dimension, q-analogues of Wang and Wang, [12] are proved. Then in Section 5, mixed formulas of the same kind are proved. Most of the proofs are similar, where different functions, previously used for the case $q = 1$, are used in each proof.

We now start with the definitions. Some of the notation is well-known and can be found in the book [3]. The variables i, j, k, l, m, n, ν will denote positive integers, and λ will denote complex numbers when nothing else is stated.

Definition 1. The Gauss q-binomial coefficient are defined by

$$\binom{n}{k}_q \equiv \frac{\{n\}_q!}{\{k\}_q!\{n-k\}_q!}, k = 0, 1, \ldots, n. \tag{1}$$

Let a and b be any elements with commutative multiplication. Then the NWA q-addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n = 0, 1, 2, \ldots. \tag{2}$$

If $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function is defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{3}$$

The following theorem shows how Ward numbers usually appear in applications.

Theorem 1.1. *Assume that $n, k \in \mathbb{N}$. Then*

$$(\overline{n}_q)^k = \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1, \ldots, m_n}_q, \tag{4}$$

where each partition of k is multiplied with its number of permutations.

The semiring of Ward numbers, $(\mathbb{N}_{\oplus_q}, \oplus_q, \odot_q)$ is defined as follows:

Definition 2. Let $(\mathbb{N}_{\oplus_q}, \oplus_q, \odot_q)$ denote the Ward numbers $\overline{k}_q,\ k \geq 0$ together with two binary operations: $\oplus_q$ is the usual Ward q-addition. The multiplication $\odot_q$ is defined as follows:

$$\overline{n}_q \odot_q \overline{m}_q \sim \overline{nm}_q, \tag{5}$$

where $\sim$ denotes the equivalence in the alphabet.

Theorem 1.2. *Functional equations for Ward numbers operating on the q-exponential function. First assume that the letters $\overline{m}_q$ and $\overline{n}_q$ are independent, i.e. come from two different functions, when operating with the functional. Then we have*

$$\mathrm{E}_q(\overline{m}_q \overline{n}_q t) = \mathrm{E}_q(\overline{mn}_q t). \tag{6}$$

Furthermore,

$$\mathrm{E}_q(\overline{jm}_q) = \mathrm{E}_q(\overline{j}_q)^m = \mathrm{E}_q(\overline{m}_q)^j = \mathrm{E}_q(\overline{n}_q \odot_q \overline{m}_q). \tag{7}$$

Proof. Formula (6) is proved as follows:

$$\mathrm{E}_q(\overline{m}_q \overline{n}_q t) = \mathrm{E}_q((1 \oplus_q 1 \oplus_q \cdots \oplus_q 1)\overline{n}_q t), \tag{8}$$

where the number of 1s to the left is m. But this means exactly $\mathrm{E}_q(\overline{n}_q t)^m$, and the result follows. □

Definition 3. The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1, \ldots, m_n$ running over all non-negative integer values.

Given an integer k, the formula

$$m_0 + m_1 + \ldots + m_j = k \tag{9}$$

determines a set $J_{m_0,\ldots,m_j} \in \mathbb{N}^{j+1}$.

Then if $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, its k'th NWA-power is given by

$$(\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{m_l \in J_{m_0,\ldots,m_j}} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q. \tag{10}$$

We will later use a similar formula when $q = 1$ for several proofs.

In order to solve systems of equations with letters as variables and Ward number coefficients, we introduce a division with a Ward number. This is equivalent to q-rational numbers with Ward numbers instead of integers.

Definition 4. Let $\mathbb{Q}_{\oplus_q}$ denote the set of objects of the following type:

$$\frac{\overline{m}_q}{\overline{n}_q}, \text{ where } \frac{\overline{m}_q}{\overline{m}_q} \equiv 1, \tag{11}$$

together with a linear functional

$$v, \ \mathbb{R}[x] \times \mathbb{Q}_{\oplus_q} \to \mathbb{R}, \tag{12}$$

called the evaluation. If $v(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$v\left(\frac{\overline{m}_q}{\overline{n}_q}\right) \equiv \sum_{k=0}^{\infty} a_k \frac{(\overline{m}_q)^k}{(\overline{n}_q)^k}. \tag{13}$$

Definition 5. For every power series $f_n(t)$, the q-Appell polynomials or Φ_q polynomials of degree ν and order n have the following generating function:

$$f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)}(x). \tag{14}$$

For $x = 0$ we get the $\Phi_{\nu,q}^{(n)}$ number of degree ν and order n.

Definition 6. For $f_n(t)$ of the form $h(t)^n$, we call the q-Appell polynomial Φ_q in (14) *multiplicative.*

Examples of multiplicative q-Appell polynomials are the two q-Appell polynomials in this article.

2. The NWA q-Apostol–Bernoulli polynomials.

Definition 7. The generalized NWA q-Apostol–Bernoulli polynomials $\mathcal{B}_{\mathrm{NWA},\lambda,\nu,q}^{(n)}(x)$ are defined by

$$\frac{t^n}{(\lambda \mathrm{E}_q(t) - 1)^n} \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{B}_{\mathrm{NWA},\lambda,\nu,q}^{(n)}(x)}{\{\nu\}_q!}, \ |t + \log \lambda| < 2\pi. \tag{15}$$

Notice that the exponent n is an integer.

Definition 8. A q-analogue of [10, (20) p. 381], the multiple q-power sum is defined by

$$s_{\mathrm{NWA},\lambda,m,q}^{(l)}(n) \equiv \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} \lambda^k \left(\overline{k_q}\right)^m, \tag{16}$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}, \ \forall j_i \geq 0$.

Definition 9. A q-analogue of [10, (46) p. 386], the multiple alternating q-power sum is defined by

$$\sigma^{(l)}_{\mathrm{NWA},\lambda,m,q}(n) \equiv (-1)^l \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} (-\lambda)^k \left(\overline{k_q}\right)^m, \tag{17}$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1},\ \forall j_i \geq 0$.

Remark 1. For $l = 1$, formulas (16) and (17) reduce to single sums, as will be seen in section 4.

We now start rather abruptly with the theorems; we note that limits like $\lambda \to 1$ and $q \to 1$ can be taken anywhere in the paper, and also in the next one [6]; see the subsequent corollaries. Much care is needed in the proofs, since the Ward numbers need careful handling.

Theorem 2.1. *A q-analogue of* [10, p. 380], *multiplication formula for q-Apostol–Bernoulli polynomials.*

$$\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(\overline{m}_q x) = \frac{(\overline{m}_q)^\nu}{(\overline{m}_q)^n} \sum_{|\vec{j}|=n} \lambda^k \binom{n}{\vec{j}} \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda^m,\nu,q}\left(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}\right), \tag{18}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, *and* $\frac{\overline{k}_q}{\overline{m}_q} \in \mathbb{Q}_{\oplus_q}$.

Proof. We use the well-known formula for a geometric sum.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(\overline{m}_q x)\frac{t^\nu}{\{\nu\}_q!} = \frac{t^n}{(\lambda \mathrm{E}_q(t)-1)^n}\mathrm{E}_q(\overline{m}_q xt) \\
&= \frac{t^n}{(\lambda^m \mathrm{E}_q(\overline{m}_q t)-1)^n}\left(\sum_{i=0}^{m-1}\lambda^i \mathrm{E}_q(\overline{i}_q t)\right)^n \mathrm{E}_q(\overline{m}_q xt) \\
&\overset{\text{by}(7)}{=} \left(\frac{t}{(\lambda^m \mathrm{E}_q(\overline{m}_q t)-1)}\right)^n \sum_{|\vec{j}|=n}\binom{n}{\vec{j}}\lambda^k \mathrm{E}_q\left((x \oplus_q \frac{\overline{k}_q}{\overline{m}_q})\overline{m}_q t\right) \\
&= \sum_{\nu=0}^{\infty}\left(\frac{(\overline{m}_q)^\nu}{(\overline{m}_q)^n}\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}\lambda^k \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda^m,\nu,q}\left(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.
\end{aligned} \tag{19}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Corollary 2.2. *A q-analogue of* [10, p. 381]*:*

$$\mathcal{B}_{\mathrm{NWA},\lambda,\nu,q}(\overline{m}_q x) = \frac{(\overline{m}_q)^\nu}{m}\sum_{j=0}^{m-1}\lambda^j \mathcal{B}_{\mathrm{NWA},\lambda^m,\nu,q}\left(x \oplus_q \frac{\overline{j}_q}{\overline{m}_q}\right). \tag{20}$$

Corollary 2.3. *A q-analogue of Carlitz formula* [2], [10, p. 381]

$$\mathcal{B}^{(n)}_{\mathrm{NWA},\nu,q}(\overline{m}_q x) = \frac{(\overline{m}_q)^{\nu}}{(\overline{m}_q)^n} \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} \mathcal{B}^{(n)}_{\mathrm{NWA},\nu,q}\left(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}\right), \tag{21}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$*, and* $\frac{\overline{k}_q}{\overline{m}_q} \in \mathbb{Q}_{\oplus_q}$.

Theorem 2.4. *A formula for a multiple q-power sum, a q-analogue of* [10, (25) p. 382]*:*

$$\begin{aligned} s^{(l)}_{\mathrm{NWA},\lambda,m,q}(n) &= \sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{l-j}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}} \\ &\times \left(\sum_{k=0}^{m+l} \binom{m+l}{k}_q \mathcal{B}^{(j)}_{\mathrm{NWA},\lambda,k,q}\left(\overline{(n-1)j+l}_q\right) \mathcal{B}^{(l-j)}_{\mathrm{NWA},\lambda,m+l-k,q} \right). \end{aligned} \tag{22}$$

Proof. We use the generating function technique. Put $k = j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$. It is assumed that $j_i \geq 0, 1 \leq i \leq n-1$, zeros are neglected.

$$\begin{aligned} &\sum_{\nu=0}^{\infty} s^{(l)}_{\mathrm{NWA},\lambda,\nu,q}(n) \frac{t^{\nu}}{\{\nu\}_q!} \overset{\text{by}(16)}{=} \sum_{\nu=0}^{\infty} \left(\sum_{|\vec{j}|=l} \binom{l}{\vec{j}} \lambda^k \left(\overline{k}_q\right)^{\nu} \right) \frac{t^{\nu}}{\{\nu\}_q!} \\ &\overset{\text{by}(16)}{=} \left(\lambda \mathrm{E}_q(t) + \lambda^2 \mathrm{E}_q(\overline{2}_q t) + \cdots + \lambda^{n-1} \mathrm{E}_q(\overline{n-1}_q t)\right)^l \\ &= \left(\frac{\lambda^n \mathrm{E}_q(\overline{n}_q t)}{\lambda \mathrm{E}_q(t) - 1} - \frac{\lambda \mathrm{E}_q(t)}{\lambda \mathrm{E}_q(t) - 1} \right)^l \\ &= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \left(\frac{\lambda^n \mathrm{E}_q(\overline{n}_q t)}{\lambda \mathrm{E}_q(t) - 1} \right)^j \left(\frac{\lambda \mathrm{E}_q(t)}{\lambda \mathrm{E}_q(t) - 1} \right)^{l-j} \\ &\overset{\text{by}(7)}{=} t^{-l} \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \lambda^{(n-1)j+l} \sum_{k=0}^{\infty} \mathcal{B}^{(j)}_{\mathrm{NWA},\lambda,k,q}\left(\overline{(n-1)j+l}_q\right) \frac{t^k}{\{k\}_q!} \\ &\times \sum_{i=0}^{\infty} \mathcal{B}^{(l-j)}_{\mathrm{NWA},\lambda,i,q} \frac{t^i}{\{i\}_q!} = \sum_{\nu=0}^{\infty} \left[\sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{l-j}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}} \right. \\ &\left. \times \sum_{k=0}^{m+l} \binom{m+l}{k}_q \mathcal{B}^{(j)}_{\mathrm{NWA},\lambda,k,q}\left(\overline{(n-1)j+l}_q\right) \mathcal{B}^{(l-j)}_{\mathrm{NWA},\lambda,m+l-k,q} \right] \frac{t^{\nu}}{\{\nu\}_q!}. \end{aligned} \tag{23}$$

The theorem follows by equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_q!}$. □

Corollary 2.5. *A q-analogue of* [10, (26) p. 382]*: The generating function for $s^{(l)}_{\mathrm{NWA},\lambda,\nu,q}(n)$ is*

$$\sum_{\nu=0}^{\infty} s^{(l)}_{\mathrm{NWA},\lambda,\nu,q}(n)\frac{t^{\nu}}{\{\nu\}_q!} = \left(\frac{\lambda^n \mathrm{E}_q(\overline{n}_q t)}{\lambda \mathrm{E}_q(t)-1} - \frac{\lambda \mathrm{E}_q(t)}{\lambda \mathrm{E}_q(t)-1}\right)^l \tag{24}$$

$$= \left(\lambda \mathrm{E}_q(t) + \lambda^2 \mathrm{E}_q(\overline{2}_q t) + \cdots + \lambda^{n-1}\mathrm{E}_q(\overline{n-1}_q t)\right)^l.$$

Theorem 2.6. *A recurrence relation for q-Apostol–Bernoulli numbers, a q-analogue of* [10, (32) p. 384].

$$(\overline{m}_q)^l \mathcal{B}^{(l)}_{\mathrm{NWA},\lambda,n,q} = \sum_{j=0}^{n}\binom{n}{j}_q \frac{(\overline{m}_q)^n}{(\overline{m}_q)^{n-j}} \mathcal{B}^{(l)}_{\mathrm{NWA},\lambda^m,j,q} s^{(l)}_{\mathrm{NWA},\lambda,n-j,q}(m), \tag{25}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$.

Proof. We use the definition of q-Appell numbers as q-Appell polynomial at $x = 0$.

$$(\overline{m}_q)^l \mathcal{B}^{(l)}_{\mathrm{NWA},\lambda,n,q} \overset{\text{by}(18)}{=} (\overline{m}_q)^n \sum_{|\vec{\nu}|=l} \lambda^k \binom{l}{\vec{\nu}} \mathcal{B}^{(l)}_{\mathrm{NWA},\lambda^m,n,q}\left(\frac{\overline{k}_q}{\overline{m}_q}\right)$$

$$= (\overline{m}_q)^n \sum_{|\vec{\nu}|=l} \lambda^k \binom{l}{\vec{\nu}} \sum_{j=0}^{n}\binom{n}{j}_q \mathcal{B}^{(l)}_{\mathrm{NWA},\lambda^m,j,q}\left(\frac{\overline{k}_q}{\overline{m}_q}\right)^{n-j} \tag{26}$$

$$= \sum_{j=0}^{n}\binom{n}{j}_q \frac{(\overline{m}_q)^n}{(\overline{m}_q)^{n-j}} \mathcal{B}^{(l)}_{\mathrm{NWA},\lambda^m,j,q} \sum_{|\vec{\nu}|=l} \lambda^k \binom{l}{\vec{\nu}} (\overline{k}_q)^{n-j} \overset{\text{by}(16)}{=} \text{LHS}.$$

□

3. The NWA q-Apostol–Euler polynomials.

We start with some repetition from [3]:

Definition 10. The generating function for the first q-Euler polynomials of degree ν and order n, $\mathrm{F}^{(n)}_{\mathrm{NWA},\nu,q}(x)$, is given by

$$\frac{2^n \mathrm{E}_q(xt)}{(\mathrm{E}_q(t)+1)^n} = \sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\mathrm{F}^{(n)}_{\mathrm{NWA},\nu,q}(x),\ |t| < \pi. \tag{27}$$

Definition 11. The generalized NWA q-Apostol–Euler polynomials $\mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(x)$ are defined by

$$\frac{2^n}{(\lambda \mathrm{E}_q(t)+1)^n}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty}\frac{t^{\nu}\mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(x)}{\{\nu\}_q!},\ |t+\log\lambda| < \pi. \tag{28}$$

Theorem 3.1. *A q-analogue of* [10, (37) p. 385], *first multiplication formula for q-Apostol–Euler polynomials.*

$$\mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(\overline{m}_q x) = (\overline{m}_q)^{\nu} \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} \mathcal{F}^{(n)}_{\mathrm{NWA},\lambda^m,\nu,q}\left(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}\right), \tag{29}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m *odd.*

Proof.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} \mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(\overline{m}_q x)\frac{t^\nu}{\{\nu\}_q!} = \frac{2^n}{(\lambda \mathrm{E}_q(t)+1)^n}\mathrm{E}_q(\overline{m}_q x t)\\
&= \frac{2^n}{(\lambda^m \mathrm{E}_q(\overline{m}_q t)+1)^n}\left(\sum_{i=0}^{m-1}(-\lambda)^i \mathrm{E}_q(\overline{i}_q t)\right)^n \mathrm{E}_q(\overline{m}_q x t)\\
&= \left(\frac{2}{(\lambda^m \mathrm{E}_q(\overline{m}_q t)+1)}\right)^n \sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k \mathrm{E}_q\left((x \oplus_q \frac{\overline{k}_q}{\overline{m}_q})\overline{m}_q t\right)\\
&= \sum_{\nu=0}^{\infty}\left((\overline{m}_q)^\nu \sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k \mathcal{F}^{(n)}_{\mathrm{NWA},\lambda^m,\nu,q}\left(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.
\end{aligned} \tag{30}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Theorem 3.2. *A q-analogue of* [10, (38) p. 385], *second multiplication formula for q-Apostol–Euler polynomials.*

$$\begin{aligned}
&\mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q}(\overline{m}_q x)\\
&= \frac{(-2)^n(\overline{m}_q)^{\nu+n}}{\{\nu+1\}_{n,q}(\overline{m}_q)^n}\sum_{|\vec{j}|=n}(-\lambda^k)\binom{n}{\vec{j}}\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda^m,\nu+n,q}\left(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}\right),
\end{aligned} \tag{31}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m *even.*

Corollary 3.3. *A q-analogue of* [10, (43) p. 386]*:*

$$\begin{aligned}
&\mathcal{F}_{\mathrm{NWA},\lambda,\nu,q}(\overline{m}_q x) =\\
&= \begin{cases} (\overline{m}_q)^\nu \sum_{j=0}^{m-1}(-\lambda)^j \mathcal{F}_{\mathrm{NWA},\lambda^m,\nu,q}\left(x \oplus_q \frac{\overline{j}_q}{\overline{m}_q}\right), & m \text{ odd },\\ \frac{-2(\overline{m}_q)^{\nu+1}}{m\{\nu+1\}_q}\sum_{j=0}^{m-1}(-\lambda)^j \mathcal{B}_{\mathrm{NWA},\lambda^m,\nu+1,q}\left(x \oplus_q \frac{\overline{j}_q}{\overline{m}_q}\right), & m \text{ even}, \end{cases}
\end{aligned} \tag{32}$$

where $\frac{\overline{j}_q}{\overline{m}_q} \in \mathbb{Q}_{\oplus_q}$.

Theorem 3.4. *A formula for a multiple alternating q-power sum, a q-analogue of* [10, (51) p. 387]*:*

$$\sigma^{(l)}_{\mathrm{NWA},\lambda,m,q}(n) = 2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}} \times\left(\sum_{k=0}^{m+l}\binom{m+l}{k}_q \mathcal{F}^{(j)}_{\mathrm{NWA},\lambda,k,q}\left(\overline{(n-1)j+l_q}\right)\mathcal{F}^{(l-j)}_{\mathrm{NWA},\lambda,n+l-k,k,q}\right). \tag{33}$$

Proof. We use the generating function technique. Put $k = j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$. It is assumed that $j_i \geq 0, 1 \leq i \leq n-1$.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q}(n)\frac{t^\nu}{\{\nu\}_q!} \overset{\text{by}(17)}{=} \sum_{\nu=0}^{\infty}\left(\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}(-1)^l(-\lambda)^k\left(\overline{k_q}\right)^\nu\right)\frac{t^\nu}{\{\nu\}_q!}\\
&\overset{\text{by}(17)}{=} (-1)^l\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}\left(-\lambda\mathrm{E}_q(t)\right)^k\\
&= \left(\lambda\mathrm{E}_q(t) - \lambda^2\mathrm{E}_q(\overline{2}_q t) + \cdots + (-1)^n\lambda^{n-1}\mathrm{E}_q(\overline{n-1}_q t))\right)^l\\
&= \left(\frac{(-\lambda)^n\mathrm{E}_q(\overline{n}_q t)}{\lambda\mathrm{E}_q(t)+1} + \frac{\lambda\mathrm{E}_q(t)}{\lambda\mathrm{E}_q(t)+1}\right)^l\\
&= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}\left(\frac{(-\lambda)^n\mathrm{E}_q(\overline{n}_q t)}{\lambda\mathrm{E}_q(t)+1}\right)^j\left(\frac{\lambda\mathrm{E}_q(t)}{\lambda\mathrm{E}_q(t)+1}\right)^{l-j}\\
&\overset{\text{by}(7)}{=} 2^{-l}\sum_{j=0}^{l}\binom{l}{j}(-1)^{jn}\lambda^{(n-1)j+l}\sum_{k=0}^{\infty}\mathcal{F}^{(j)}_{\mathrm{NWA},\lambda,k,q}\left(\overline{(n-1)j+l_q}\right)\frac{t^k}{\{k\}_q!}\\
&\times\sum_{i=0}^{\infty}\mathcal{F}^{(l-j)}_{\mathrm{NWA},\lambda,i,q}\frac{t^i}{\{i\}_q!} = \sum_{\nu=0}^{\infty}\left[2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}}\right.\\
&\left.\times\sum_{k=0}^{m+l}\binom{m+l}{k}_q\mathcal{F}^{(j)}_{\mathrm{NWA},\lambda,k,q}\left(\overline{(n-1)j+l_q}\right)\mathcal{F}^{(l-j)}_{\mathrm{NWA},\lambda,n+l-k,k,q}\right]\frac{t^\nu}{\{\nu\}_q!}.
\end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Corollary 3.5. *A q-analogue of* [10, (52) p. 387]*: The generating function for* $\sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q}(n)$ *is*

$$\begin{aligned}
\sum_{\nu=0}^{\infty}\sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q}(n)\frac{t^\nu}{\{\nu\}_q!} &= \left(\frac{(-\lambda)^n\mathrm{E}_q(\overline{n}_q t)}{\lambda\mathrm{E}_q(t)-1} + \frac{\lambda\mathrm{E}_q(t)}{\lambda\mathrm{E}_q(t)+1}\right)^l\\
&= \left(\lambda\mathrm{E}_q(t) - \lambda^2\mathrm{E}_q(\overline{2}_q t) + \cdots + (-1)^n\lambda^{n-1}\mathrm{E}_q(\overline{n-1}_q t)\right)^l.
\end{aligned} \tag{34}$$

Theorem 3.6. *A q-analogue of* [10, p. 389]. *For m odd, we have the following recurrence relation for q-Apostol–Euler numbers.*

$$\mathcal{F}^{(l)}_{\mathrm{NWA},\lambda,n,q} = (-1)^l \sum_{j=0}^{n} \binom{n}{j}_q \frac{(\overline{m}_q)^n}{(\overline{m}_q)^{n-j}} \mathcal{F}^{(l)}_{\mathrm{NWA},\lambda^m,j,q} \sigma^{(l)}_{\mathrm{NWA},\lambda,n-j,q}(m), \tag{35}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$.

Proof.

$$\mathcal{F}^{(l)}_{\mathrm{NWA},\lambda,n,q} \overset{\mathrm{by}(29)}{=} (\overline{m}_q)^n \sum_{|\vec{\nu}|=l} (-\lambda)^k \binom{l}{\vec{\nu}} \mathcal{F}^{(l)}_{\mathrm{NWA},\lambda^m,n,q}\left(\frac{\overline{k}_q}{\overline{m}_q}\right)$$

$$= (\overline{m}_q)^n \sum_{|\vec{\nu}|=l} (-\lambda)^k \binom{l}{\vec{\nu}} \sum_{j=0}^{n} \binom{n}{j}_q \mathcal{F}^{(l)}_{\mathrm{NWA},\lambda^m,j,q}\left(\frac{\overline{k}_q}{\overline{m}_q}\right)^{n-j} \tag{36}$$

$$= \sum_{j=0}^{n} \binom{n}{j}_q \frac{(\overline{m}_q)^n}{(\overline{m}_q)^{n-j}} \mathcal{F}^{(l)}_{\mathrm{NWA},\lambda^m,j,q} \sum_{|\vec{\nu}|=l} (-\lambda)^k \binom{l}{\vec{\nu}}_q (\overline{k}_q)^{n-j} \overset{\mathrm{by}(17)}{=} \mathrm{LHS}.$$

□

4. Single formulas for Apostol *q*-power sums. In order to keep the same notation as in [3], we make a slight change from [12, p. 309]. The following definitions are special cases of the q-power sums in section 2.

Definition 12. Almost a q-analogue of [12, p. 309], the q-power sum and the alternate q-power sum (with respect to λ), are defined by

$$s_{\mathrm{NWA},\lambda,m,q}(n) \equiv \sum_{k=0}^{n-1} \lambda^k (\overline{k}_q)^m \text{ and } \sigma_{\mathrm{NWA},\lambda,m,q}(n) \equiv \sum_{k=0}^{n-1} (-1)^k \lambda^k (\overline{k}_q)^m. \tag{37}$$

Their respective generating functions are

$$\sum_{m=0}^{\infty} s_{\mathrm{NWA},\lambda,m,q}(n) \frac{t^m}{\{m\}_q!} = \frac{\lambda^n \mathrm{E}_q(\overline{n}_q t) - 1}{\lambda \mathrm{E}_q(t) - 1} \tag{38}$$

and

$$\sum_{m=0}^{\infty} \sigma_{\mathrm{NWA},\lambda,m,q}(n) \frac{t^m}{\{m\}_q!} = \frac{(-1)^{n+1} \lambda^n \mathrm{E}_q(\overline{n}_q t) + 1}{\lambda \mathrm{E}_q(t) + 1}. \tag{39}$$

Proof. Let us prove (38). We have

$$\sum_{m=0}^{\infty} s_{\mathrm{NWA},\lambda,m,q}(n) \frac{t^m}{\{m\}_q!} = \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \lambda^k \frac{(\overline{k}_q t)^m}{\{m\}_q!} \overset{\mathrm{by}(6)}{=} \sum_{k=0}^{n-1} \lambda^k (\mathrm{E}_q(t))^k = \mathrm{RHS}.$$

□

We have the following special cases:

$$s_{\mathrm{NWA},\lambda,m,q}(1)=\sigma_{\mathrm{NWA},\lambda,m,q}(1)=\delta_{0,m}, \tag{40}$$

$$s_{\mathrm{NWA},\lambda,m,q}(2)=\delta_{0,m}+\lambda,\ \sigma_{\mathrm{NWA},\lambda,m,q}(2)=\delta_{0,m}-\lambda. \tag{41}$$

Theorem 4.1. *A q-analogue of* [12, p. 310]*, and extensions of* [3, p. 121, 131]*:*

$$s_{\mathrm{NWA},\lambda,m,q}(n)=\frac{\lambda^n\mathcal{B}_{\mathrm{NWA},\lambda,m+1,q}(\overline{n}_q)-\mathcal{B}_{\mathrm{NWA},\lambda,m+1,q}}{\{m+1\}_q}. \tag{42}$$

$$\sigma_{\mathrm{NWA},\lambda,m,q}(n)=\frac{(-1)^{n+1}\lambda^n\mathcal{F}_{\mathrm{NWA},\lambda,m,q}(\overline{n}_q)-\mathcal{F}_{\mathrm{NWA},\lambda,m,q}}{2} \tag{43}$$

Theorem 4.2. *A q-analogue of* [12, (18), p. 311],

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}_q\frac{(\bar{i}_q)^k}{i}(\bar{j}_q)^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q}\left(\bar{j}_qx\right)s_{\mathrm{NWA},\lambda^j,n-k,q}(i)\\&=\sum_{k=0}^{n}\binom{n}{k}_q\frac{(\bar{j}_q)^k}{j}(\bar{i}_q)^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^j,k,q}\left(\bar{i}_qx\right)s_{\mathrm{NWA},\lambda^i,n-k,q}(j)\\&=\frac{(\bar{i}_q)^n}{i}\sum_{m=0}^{i-1}\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^i,n,q}\left(\bar{j}_qx\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\\&=\frac{(\bar{j}_q)^n}{j}\sum_{m=0}^{j-1}\lambda^{im}\mathcal{B}_{\mathrm{NWA},\lambda^j,n,q}\left(\bar{i}_qx\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right).\end{aligned} \tag{44}$$

Proof. Define the following function, symmetric in i and j.

$$\begin{aligned}f_q(t)&\equiv\frac{t\mathrm{E}_q(\overline{ij}_qxt)(\lambda^{ij}\mathrm{E}_q(\overline{ij}_qt)-1)}{(\lambda^i\mathrm{E}_q(\bar{i}_qt)-1)(\lambda^j\mathrm{E}_q(\bar{j}_qt)-1)}\\&=\left(\frac{(\bar{i}_qt)^1\mathrm{E}_q(\overline{ij}_qxt)}{\lambda^i\mathrm{E}_q(\bar{i}_qt)-1}\right)\left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_qt)-1}{\lambda^j\mathrm{E}_q(\bar{j}_qt)-1}\right)\frac{1}{i}.\end{aligned} \tag{45}$$

By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$\begin{aligned}f_q(t)&=\left(\sum_{\nu=0}^{\infty}\mathcal{B}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_qx\right)\frac{(\bar{i}_qt)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^j,m,q}(i)\frac{(\bar{j}_qt)^m}{\{m\}_q!}\right)\frac{1}{i}\\&=\frac{\left(\bar{i}_q\right)^1t}{\lambda^i\mathrm{E}_q(\bar{i}_qt)-1}\sum_{m=0}^{i-1}\lambda^{jm}\left(\mathrm{E}_q\left(\bar{j}_qx\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\bar{i}_qt\right)\frac{1}{i}\\&=\sum_{\nu=0}^{\infty}\left(\frac{(\bar{i}_q)^\nu}{i}\sum_{m=0}^{i-1}\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_qx\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.\end{aligned} \tag{46}$$

The theorem follows by equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_q!}$ and using the symmetry in i and j of $f_q(t)$. □

Corollary 4.3. *A q-analogue of* [12, (19), p. 311],

$$\begin{aligned}\mathcal{B}_{\mathrm{NWA},\lambda,n,q}\left(\bar{i}_q x\right) &= \sum_{k=0}^{n}\binom{n}{k}_q \frac{(\bar{i}_q)^k}{i}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q}\left(x\right) s_{\mathrm{NWA},\lambda,n-k,q}(i)\\ &= \frac{(\bar{i}_q)^n}{i}\sum_{m=0}^{i-1}\lambda^m \mathcal{B}_{\mathrm{NWA},\lambda^i,n,q}\left(x \oplus_q \frac{\overline{m}_q}{\bar{i}_q}\right).\end{aligned} \tag{47}$$

Proof. Put $j = 1$ in (44) and use (41). □

Remark 2. This proves formula (20) again.

Corollary 4.4. *A q-analogue of* [12, (20), p. 311],

$$\begin{aligned}&\sum_{m=0}^{1}\lambda^{im}\mathcal{B}_{\mathrm{NWA},\lambda^2,n,q}\left(\bar{i}_q x \oplus_q \frac{\overline{im}_q}{\bar{2}_q}\right)\\ &= \frac{2}{(\bar{2}_q)^n}\sum_{k=0}^{n}\binom{n}{k}_q \frac{(\bar{i}_q)^k}{i}(\bar{2}_q)^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q}\left(\bar{2}_q x\right) s_{\mathrm{NWA},\lambda^2,n-k,q}(i)\\ &= \frac{2}{(\bar{2}_q)^n}\frac{(\bar{i}_q)^n}{i}\sum_{m=0}^{i-1}\lambda^{2m}\mathcal{B}_{\mathrm{NWA},\lambda^i,n,q}\left(\bar{2}_q x \oplus_q \frac{\overline{2m}_q}{\bar{i}_q}\right).\end{aligned} \tag{48}$$

Proof. Put $j = 2$ in (44) and multiply by $\frac{2}{(\bar{2}_q)^n}$. □

Moreover, we have

$$\mathcal{B}_{\mathrm{NWA},\lambda,n,q}\left(x\right) = \frac{(\bar{2}_q)^n}{2}\sum_{m=0}^{1}\lambda^m \mathcal{B}_{\mathrm{NWA},\lambda^2,n,q}\left(\frac{x}{\bar{2}_q} \oplus_q \frac{\overline{m}_q}{\bar{2}_q}\right). \tag{49}$$

Proof. Put $i = 2$ in (47) and replace x by $\dfrac{x}{\bar{2}_q}$. □

For $\lambda = 1$ and $x = 0$, this reduces to

$$\mathrm{B}_{\mathrm{NWA},n,q}\left(\frac{1}{\bar{2}_q}\right) = \left(\frac{2}{(\bar{2}_q)^n} - 1\right)\mathrm{B}_{\mathrm{NWA},n,q}. \tag{50}$$

Theorem 4.5. *A q-analogue of* [12, (22) p. 312]. *Assume that i and j are either both odd, or both even, then we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}_q(\bar{i}_q)^k(\bar{j}_q)^{n-k}\mathcal{F}_{\mathrm{NWA},\lambda^i,k,q}\left(\bar{j}_qx\right)\sigma_{\mathrm{NWA},\lambda^j,n-k,q}(i)\\
&=\sum_{k=0}^{n}\binom{n}{k}_q(\bar{j}_q)^k(\bar{i}_q)^{n-k}\mathcal{F}_{\mathrm{NWA},\lambda^j,k,q}\left(\bar{i}_qx\right)\sigma_{\mathrm{NWA},\lambda^j,n-k,q}(i)\\
&=(\bar{i}_q)^n\sum_{m=0}^{i-1}\lambda^{jm}(-1)^m\mathcal{F}_{\mathrm{NWA},\lambda^i,n,q}\left(\bar{j}_qx\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\\
&=(\bar{j}_q)^n\sum_{m=0}^{j-1}\lambda^{im}(-1)^m\mathcal{F}_{\mathrm{NWA},\lambda^j,n,q}\left(\bar{i}_qx\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right).
\end{aligned}\tag{51}$$

Proof. Define the following symmetric function

$$\begin{aligned}
f_q(t)&\equiv\frac{\mathrm{E}_q(\overline{ij}_qxt)((-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_qt)+1)}{(\lambda^i\mathrm{E}_q(\bar{i}_qt)+1)(\lambda^j\mathrm{E}_q(\bar{j}_qt)+1)}\\
&=\frac{1}{2}\left(\frac{2\mathrm{E}_q(\overline{ij}_qxt)}{\lambda^i\mathrm{E}_q(\bar{i}_qt)+1}\right)\left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_qt)+1}{\lambda^j\mathrm{E}_q(\bar{j}_qt)+1}\right).
\end{aligned}\tag{52}$$

By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$\begin{aligned}
f_q(t)&=\frac{1}{2}\left(\sum_{\nu=0}^{\infty}\mathcal{F}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_qx\right)\frac{(\bar{i}_qt)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}\sigma_{\mathrm{NWA},\lambda^j,m,q}(i)\frac{(\bar{j}_qt)^m}{\{m\}_q!}\right)\\
&=\frac{1}{\lambda^i\mathrm{E}_q(\bar{i}_qt)+1}\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\mathrm{E}_q\left(\left(\bar{j}_qx\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\bar{i}_qt\right)\\
&=\frac{1}{2}\sum_{\nu=0}^{\infty}\left((\bar{i}_q)^\nu\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\mathcal{F}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\bar{j}_qx\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.
\end{aligned}\tag{53}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$ and using the symmetry in i and j of $f_q(t)$. □

Theorem 4.6. *(A q-analogue of* [12, (24) p. 313]*) For i odd we have*

$$\begin{aligned}
\mathcal{F}_{\mathrm{NWA},\lambda,n,q}\left(\bar{i}_qx\right)&=\sum_{k=0}^{n}\binom{n}{k}_q(\bar{i}_q)^k\mathcal{F}_{\mathrm{NWA},\lambda^i,k,q}\left(x\right)\sigma_{\mathrm{NWA},\lambda,n-k,q}(i)\\
&=(\bar{i}_q)^n\sum_{m=0}^{i-1}(-\lambda)^m\mathcal{F}_{\mathrm{NWA},\lambda^i,n,q}\left(x\oplus_q\frac{\overline{m}_q}{\bar{i}_q}\right).
\end{aligned}\tag{54}$$

(A q-analogue of [12, (25) p. 313]*) For i even,*

$$
\begin{aligned}
&\sum_{m=0}^{1}\lambda^{im}(-1)^m\mathcal{F}_{\mathrm{NWA},\lambda^2,n,q}\left(\bar{i}_q x\oplus_q\frac{\overline{im}_q}{\overline{2}_q}\right)\\
&=\frac{1}{(\overline{2}_q)^n}\sum_{k=0}^{n}\binom{n}{k}_q(\bar{i}_q)^k(\overline{2}_q)^{n-k}\mathcal{F}_{\mathrm{NWA},\lambda^i,k,q}\left(\overline{2}_q x\right)\sigma_{\mathrm{NWA},\lambda^2,n-k,q}(i)\\
&=\frac{(\bar{i}_q)^n}{(\overline{2}_q)^n}\sum_{m=0}^{i-1}(-1)^m\lambda^{2m}\mathcal{F}_{\mathrm{NWA},\lambda^i,n,q}\left(\overline{2}_q x\oplus_q\frac{\overline{2m}_q}{\bar{i}_q}\right).
\end{aligned}\tag{55}
$$

Proof. Put $j=1$ or 2 in (51), and divide by $(\overline{2}_q)^n$. □

Remark 3. This proves the first part of formula (32) again.

5. Apostol *q*-power sums, mixed formulas. We now turn to mixed formulas, which contain polynomials of both kinds.

Theorem 5.1. *A q-analogue of* [12, (26) p. 313]. *If i is even then*

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}_q\frac{(\bar{i}_q)^k}{i}(\bar{j}_q)^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q}\left(\bar{j}_q x\right)\sigma_{\mathrm{NWA},\lambda^j,n-k,q}(i)\\
&=-\frac{\{n\}_q}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}_q(\bar{j}_q)^k(\bar{i}_q)^{n-k-1}\\
&\quad\times\mathcal{F}_{\mathrm{NWA},\lambda^j,k,q}\left(\bar{i}_q x\right)s_{\mathrm{NWA},\lambda^i,n-k-1,q}(j)\\
&=\frac{(\bar{i}_q)^n}{i}\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^i,n,q}\left(\bar{j}_q x\oplus_q\frac{\overline{jm}_q}{\bar{i}_q}\right)\\
&=-\frac{\{n\}_q}{2}(\bar{j}_q)^{n-1}\sum_{m=0}^{j-1}\lambda^{im}\mathcal{F}_{\mathrm{NWA},\lambda^j,n-1,q}\left(\bar{i}_q x\oplus_q\frac{\overline{im}_q}{\bar{j}_q}\right).
\end{aligned}\tag{56}
$$

Proof. Define the following function

$$
\begin{aligned}
f_q(t)&\equiv\frac{t\mathrm{E}_q(\overline{ij}_q xt)((-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)+1)}{(\lambda^i\mathrm{E}_q(\bar{i}_q t)-1)(\lambda^j\mathrm{E}_q(\bar{j}_q t)+1)}\\
&=\left(\frac{(\bar{i}_q t)^1\mathrm{E}_q(\overline{ij}_q xt)}{\lambda^i\mathrm{E}_q(\bar{i}_q t)-1}\right)\left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t)+1}{\lambda^j\mathrm{E}_q(\bar{j}_q t)+1}\right)\frac{1}{i}.
\end{aligned}\tag{57}
$$

By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$f_q(t) = \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\overline{j}_q x\right) \frac{(\overline{i}_q t)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty} \sigma_{\mathrm{NWA},\lambda^j,m,q}(i) \frac{(\overline{j}_q t)^m}{\{m\}_q!}\right)\frac{1}{i}$$

$$(58) \quad = \frac{\left(\overline{i}_q\right)^1 t}{\lambda^i \mathrm{E}_q(\overline{i}_q t) - 1} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathrm{E}_q\left(\left(\overline{j}_q x \oplus_q \frac{\overline{jm}_q}{\overline{i}_q}\right)\overline{i}_q t\right)\frac{1}{i}$$

$$= \sum_{\nu=0}^{\infty}\left(\frac{(\overline{i}_q)^\nu}{i}\sum_{m=0}^{i-1}(-1)^m \lambda^{jm} \mathcal{B}_{\mathrm{NWA},\lambda^i,\nu,q}\left(\overline{j}_q x \oplus_q \frac{\overline{jm}_q}{\overline{i}_q}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.$$

By equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$, we obtain rows 1 and 3 of formula (56).

On the other hand, we can rewrite $f_q(t)$ in the following way:

$$(59) \quad \begin{aligned} f_q(t) &= -\frac{t}{2}\frac{2\mathrm{E}_q(\overline{ij}_q xt)(\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t) - 1)}{(\lambda^i \mathrm{E}_q(\overline{i}_q t) - 1)(\lambda^j \mathrm{E}_q(\overline{j}_q t) + 1)} \\ &= -\frac{t}{2}\left(\frac{2\mathrm{E}_q(\overline{ij}_q xt)}{\lambda^j \mathrm{E}_q(\overline{j}_q t) + 1}\right)\left(\frac{\lambda^{ij}\mathrm{E}_q(\overline{ij}_q t) - 1}{\lambda^i \mathrm{E}_q(\overline{i}_q t) - 1}\right). \end{aligned}$$

By using the formula for a geometric sequence, we can expand (59) in two ways:

$$f_q(t) = -\frac{t}{2}\left(\sum_{\nu=0}^{\infty} \mathcal{F}_{\mathrm{NWA},\lambda^j,\nu,q}(\overline{i}_q x) \frac{(\overline{j}_q t)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty} s_{\mathrm{NWA},\lambda^i,m,q}(j) \frac{(\overline{i}_q t)^m}{\{m\}_q!}\right)$$

$$(60) \quad = -\frac{t}{2}\sum_{m=0}^{j-1} \lambda^{im} \frac{2}{\lambda^j \mathrm{E}_q(\overline{j}_q t) + 1} \mathrm{E}_q\left(\left(\overline{i}_q x \oplus_q \frac{\overline{im}_q}{\overline{j}_q}\right)\overline{j}_q t\right)$$

$$= -\frac{t}{2}\sum_{\nu=0}^{\infty}\left((\overline{j}_q)^\nu \sum_{m=0}^{j-1} \lambda^{im} \mathcal{F}_{\mathrm{NWA},\lambda^j,\nu,q}\left(\overline{i}_q x \oplus_q \frac{\overline{im}_q}{\overline{j}_q}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.$$

By equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$, we obtain rows 2 and 4 of formula (56). □

Corollary 5.2. *A q-analogue of* [12, (28) p. 313]. *If i is even, then*

$$(61) \quad \begin{aligned} &\mathcal{F}_{\mathrm{NWA},\lambda,n-1,q}\left(\overline{i}_q x\right) \\ &= -\frac{2}{\{n\}_q}\sum_{k=0}^{n}\binom{n}{k}_q \frac{(\overline{i}_q)^k}{i}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q}(x)\, \sigma_{\mathrm{NWA},\lambda,n-k,q}(i) \\ &= -\frac{2(\overline{i}_q)^n}{i\{n\}_q}\sum_{m=0}^{i-1}(-\lambda)^m \mathcal{B}_{\mathrm{NWA},\lambda^i,n,q}\left(x \oplus_q \frac{\overline{m}_q}{\overline{i}_q}\right). \end{aligned}$$

Proof. Put $j = 1$ in formula (56) and multiply by $-\frac{2}{\{n\}_q}$. □

Corollary 5.3. *A q-analogue of* [12, (29) p. 313].

$$
\begin{aligned}
&\mathcal{F}_{\mathrm{NWA},\lambda,n-1,q}(x) \\
&= -\frac{2}{\{n\}_q}\sum_{k=0}^{n}\binom{n}{k}_q \frac{(\overline{2}_q)^k}{2}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q}\left(\frac{x}{\overline{2}_q}\right)\sigma_{\mathrm{NWA},\lambda,n-k,q}(2) \\
&= -\frac{(\overline{2}_q)^n}{\{n\}_q}\sum_{m=0}^{1}(-\lambda)^m\mathcal{B}_{\mathrm{NWA},\lambda^2,n,q}\left(\frac{x}{\overline{2}_q}\oplus_q \frac{\overline{m}_q}{\overline{2}_q}\right).
\end{aligned}
\tag{62}
$$

Proof. Put $i = 2$ in formula (61), and replace x by $\frac{x}{\overline{2}_q}$. □

Corollary 5.4. *A q-analogue of* [12, (31) p. 314]. *If i is even, then*

(63)

$$
\begin{aligned}
&\sum_{m=0}^{1}\lambda^{im}\mathcal{F}_{\mathrm{NWA},\lambda^2,n-1,q}\left(\bar{i}_q x\oplus_q \frac{\overline{im}_q}{\overline{2}_q}\right) \\
&= -\frac{2}{\{n\}_q(\overline{2}_q)^{n-1}}\sum_{k=0}^{n}\binom{n}{k}_q \frac{(\bar{i}_q)^k}{i}(\overline{2}_q)^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q}\big(\overline{2}_q x\big)\sigma_{\mathrm{NWA},\lambda^2,n-k,q}(i) \\
&= \frac{1}{(\overline{2}_q)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}_q(\overline{2}_q)^k(\bar{i}_q)^{n-k-1}\mathcal{F}_{\mathrm{NWA},\lambda^2,k,q}\big(\bar{i}_q x\big)s_{\mathrm{NWA},\lambda^i,n-k-1,q}(2) \\
&= -\frac{2}{\{n\}_q(\overline{2}_q)^{n-1}}\frac{(\bar{i}_q)^n}{i}\sum_{m=0}^{i-1}(-1)^m\lambda^{2m}\mathcal{B}_{\mathrm{NWA},\lambda^i,n,q}\left(\overline{2}_q x\oplus_q \frac{\overline{2m}_q}{\bar{i}_q}\right).
\end{aligned}
$$

Proof. Put $j = 2$ in formula (56) and multiply by $-\frac{2}{\{n\}_q(\overline{2}_q)^{n-1}}$. □

Corollary 5.5. *A q-analogue of* [12, (32) p. 314].

$$
\begin{aligned}
&\sum_{m=0}^{1}(-1)^{m+1}\lambda^m\mathcal{B}_{\mathrm{NWA},\lambda,n,q}\left(x\oplus_q \frac{\overline{2m}_q}{\overline{2}_q}\right) \\
&= \frac{\{n\}_q(\overline{2}_q)^{n-1}}{(\overline{2}_q)^n}\sum_{m=0}^{1}\lambda^m\mathcal{F}_{\mathrm{NWA},\lambda,n-1,q}\left(x\oplus_q \frac{\overline{2m}_q}{\overline{2}_q}\right).
\end{aligned}
\tag{64}
$$

Proof. Put $i = 2$ in formula (63), replace x and λ^2 by $\dfrac{x}{\overline{2}_q}$ and λ, and multiply by $\frac{\{n\}_q(\overline{2}_q)^{n-1}}{(\overline{2}_q)^n}$. □

Corollary 5.6. *A q-analogue of* [12, (33) p. 314].

$$\sum_{m=0}^{1}(-1)^m\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^2,n,q}\left(\overline{j}_q x\oplus_q\frac{\overline{jm_q}}{\overline{2}_q}\right)$$

$$=-\frac{\{n\}_q}{(\overline{2}_q)^n}\sum_{k=0}^{n-1}\binom{n-1}{k}_q(\overline{j}_q)^k(\overline{2}_q)^{n-k-1}\mathcal{F}_{\mathrm{NWA},\lambda^j,k,q}\left(\overline{2}_q x\right)s_{\mathrm{NWA},\lambda^2,n-k-1,q}(j)$$

$$=-\frac{\{n\}_q}{(\overline{2}_q)^n}(\overline{j}_q)^{n-1}\sum_{m=0}^{j-1}\lambda^{2m}\mathcal{F}_{\mathrm{NWA},\lambda^j,n-1,q}\left(\overline{2}_q x\oplus_q\frac{\overline{2m_q}}{\overline{j}_q}\right).$$

Proof. Put $i=2$ in formula (56) and multiply by $\frac{2}{(\overline{2}_q)^n}$. □

6. Discussion. As was indicated in [5], we have considered q-analogues of the currently most popular Appell polynomials, together with corresponding power sums. The beautiful symmetry of the formulas comes from the ring structure of the q-Appell polynomials. We have not considered JHC q-Appell polynomials, since we are looking for maximal symmetry in the formulas. The q-Taylor formulas have not been used in the proofs, since the generating functions were mostly used. In a further paper [6], we will find similar expansion formulas for q-Appell polynomials of arbitrary order.

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