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General Lebesgue integral inequalities of Jensen and Ostrowski type for differentiable functions whose derivatives in absolute value are *h*-convex and applications

Abstract. Some inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral of differentiable functions whose derivatives in absolute value are h-convex are obtained. Applications for f-divergence measure are provided as well.

1. Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)| \, d\mu(t) < \infty \right\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t) \, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [37] the following result:

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Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:*

$$
\begin{aligned}
0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\
&\leq \int_\Omega f \cdot \left(\Phi' \circ f\right) d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu.
\end{aligned} \tag{1.1}
$$

In the case of discrete measure, we have:

Corollary 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$
\begin{aligned}
0 &\leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \\
&\leq \sum_{i=1}^n w_i \Phi'(x_i) x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|.
\end{aligned} \tag{1.2}
$$

Remark 1. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [49].

If $f, g : \Omega \to \mathbb{R}$ are μ-measurable functions and $f,\ g,\ fg \in L(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$
T(f, g) := \int_\Omega fg d\mu - \int_\Omega f d\mu \int_\Omega g d\mu. \tag{1.3}
$$

The following result is known in the literature as the *Grüss inequality*

$$
|T(f, g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.4}
$$

provided

$$
-\infty < \gamma \leq f(t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(t) \leq \Delta < \infty \tag{1.5}
$$

for μ-a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f(t) \leq \Gamma < \infty$ for μ-a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$\int_{\Omega}\left|f-\int_{\Omega}fd\mu\right|d\mu\leq\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]^{\frac{1}{2}}\leq\frac{1}{2}\left(\Gamma-\gamma\right). \tag{1.6}$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$\begin{aligned}
0 &\leq \int_{\Omega}\Phi\circ fd\mu-\Phi\left(\int_{\Omega}fd\mu\right)\\
&\leq \int_{\Omega}f\cdot\left(\Phi'\circ f\right)d\mu-\int_{\Omega}\Phi'\circ fd\mu\int_{\Omega}fd\mu\\
&\leq \frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\int_{\Omega}\left|f-\int_{\Omega}fd\mu\right|d\mu\\
&\leq \frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]^{\frac{1}{2}}\\
&\leq \frac{1}{4}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\left(M-m\right),
\end{aligned} \tag{1.7}$$

provided that $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ f\cdot(\Phi'\circ f)\in L(\Omega,\mu)$, with $\int_{\Omega}d\mu=1$.

The following reverse of the Jensen's inequality also holds [41].

Theorem 2. *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I. If $f:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\leq f(t)\leq M<\infty \text{ for } \mu\text{-a.e. } t\in\Omega$$

and such that $f,\ \Phi\circ f\in L(\Omega,\mu)$, then

$$\begin{aligned}
0 &\leq \int_{\Omega}\Phi\circ fd\mu-\Phi\left(\int_{\Omega}fd\mu\right)\\
&\leq \left(M-\int_{\Omega}fd\mu\right)\left(\int_{\Omega}fd\mu-m\right)\frac{\Phi'_{-}\left(M\right)-\Phi'_{+}\left(m\right)}{M-m}\\
&\leq \frac{1}{4}\left(M-m\right)\left[\Phi'_{-}\left(M\right)-\Phi'_{+}\left(m\right)\right],
\end{aligned} \tag{1.8}$$

where Φ'_{-} is the left and Φ'_{+} is the right derivative of the convex function Φ.

For other reverse of Jensen inequality and applications to divergence measures see [41].

In 1938, A. Ostrowski [80], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_{a}^{b}\Phi(t)\,dt$ and the value $\Phi(x)$, $x\in[a,b]$.

For various results related to Ostrowski's inequality see [13]–[16], [23]–[60], [64] and the references therein.

Theorem 3. *Let $\Phi : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $\Phi' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|\Phi'\|_\infty := \sup_{t\in(a,b)} |\Phi'(t)| < \infty$. Then*

$$\left|\Phi(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] \left\|\Phi'\right\|_\infty (b-a), \tag{1.9}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [45]:

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} \mid \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \ge 0 \text{ for a.e. } t \in [a,b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} \mid \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \le \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a,b] \right\}.$$

The following representation result may be stated [45].

Proposition 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{1.10}$$

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$, we have*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma,\Gamma) = \{ f : [a,b] \to \mathbb{C} \mid & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \ge 0 \\ & \textit{for a.e. } t \in [a,b] \}. \end{aligned} \tag{1.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \ge \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \ge \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma,\Gamma) := \{ f : [a,b] \to \mathbb{C} \mid & \operatorname{Re}(\Gamma) \ge \operatorname{Re} f(t) \ge \operatorname{Re}(\gamma) \text{ and} \\ & \operatorname{Im}(\Gamma) \ge \operatorname{Im} f(t) \ge \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a,b] \}. \end{aligned} \tag{1.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \ne \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{1.13}$$

The following result holds [45].

Theorem 4. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous function on $[a,b] \subset \mathring{I}$, the interior of I. For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $\Phi' \in \bar{U}_{[a,b]}(\gamma,\Gamma)$ $(= \bar{\Delta}_{[a,b]}(\gamma,\Gamma))$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\ g \in L(\Omega,\mu)$, then we have the inequality*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right)\right| \leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega |g-x|\, d\mu \tag{1.14}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned}\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu \\ &\leq \frac{1}{4}(b-a)|\Gamma-\gamma|\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left|g - \int_\Omega g d\mu\right| d\mu \\ &\leq \frac{1}{2}|\Gamma-\gamma| \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)|\Gamma-\gamma|.\end{aligned} \tag{1.16}$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right|,\ x \in [a,b],$$

under various assumptions on the absolutely continuous function Φ, which in the particular case of $x = \int_\Omega g d\mu$ provides some results connected with Jensen's inequality while in the general case provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. Preliminary Facts.

2.1. Some Identities. The following result holds [45].

Lemma 1. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous function on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\ g \in L(\Omega,\mu)$, then we have the equality*

$$\begin{aligned}\int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_\Omega g d\mu - x \right) \\ = \int_\Omega \left[(g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \lambda \right) ds \right] d\mu \end{aligned} \tag{2.1}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) = \int_\Omega \left[(g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu, \tag{2.2}$$

for any $x \in [a,b]$.

Remark 2. With the assumptions of Lemma 1 we have

$$\begin{aligned}\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) \\ = \int_\Omega \left[\left(g - \frac{a+b}{2}\right) \int_0^1 \Phi'\left((1-s)\frac{a+b}{2} + sg\right) ds \right] d\mu. \end{aligned} \tag{2.3}$$

Corollary 3. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right) \\ = \int_\Omega \left[\left(g - \int_\Omega g d\mu\right) \int_0^1 \Phi'\left((1-s)\int_\Omega g d\mu + sg\right) ds \right] d\mu. \end{aligned} \tag{2.4}$$

Proof. We observe that since $g : \Omega \to [a,b]$ and $\int_\Omega d\mu = 1$, then $\int_\Omega g d\mu \in [a,b]$ and by taking $x = \int_\Omega g d\mu$ in (2.2) we get (2.4). □

Corollary 4. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}\int_\Omega \Phi \circ g d\mu - \frac{1}{b-a}\int_a^b \Phi(x) dx - \lambda \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) \\ = \int_\Omega \left\{ \frac{1}{b-a} \int_a^b \left[(g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \lambda \right) ds \right] dx \right\} d\mu. \end{aligned} \tag{2.5}$$

Proof. Follows by integrating the identity (2.1) over $x \in [a,b]$, dividing by $b-a > 0$ and using Fubini's theorem. □

Corollary 5. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g,h : \Omega \to [a,b]$ are Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $\Phi \circ h$, g, $h \in L(\Omega,\mu)$, then we have the equality*

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \int_\Omega \Phi \circ h d\mu - \lambda \left(\int_\Omega g d\mu - \int_\Omega h d\mu \right) \\ &= \int_\Omega \int_\Omega \left[(g(t) - h(\tau)) \int_0^1 \left(\Phi' \left((1-s)\, h(\tau) + s g(t) \right) - \lambda \right) ds \right] d\mu(t) d\mu(\tau) \end{aligned} \tag{2.6}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \int_\Omega \Phi \circ h d\mu \\ &= \int_\Omega \int_\Omega \left[(g(t) - h(\tau)) \int_0^1 \Phi'((1-s)h(\tau) + s g(t)) ds \right] d\mu(t) d\mu(\tau), \end{aligned} \tag{2.7}$$

for any $x \in [a,b]$.

Remark 3. The above inequality (2.6) can be extended for two measures as follows

$$\begin{aligned} &\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left(\int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right) \\ &= \int_{\Omega_1} \int_{\Omega_2} \left[(g(t) - h(\tau)) \int_0^1 \left(\Phi'((1-s)h(\tau) + s g(t)) - \lambda \right) ds \right] d\mu_1(t) d\mu_2(\tau), \end{aligned} \tag{2.8}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$ and provided that $\Phi \circ g$, $g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

Remark 4. If $w \geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega w d\mu > 0$, then by replacing $d\mu$ with $\frac{w d\mu}{\int_\Omega w d\mu}$ in (2.1) we have the weighted equality

$$\begin{aligned} &\frac{1}{\int_\Omega w d\mu} \int_\Omega w \left(\Phi \circ g \right) d\mu - \Phi(x) - \lambda \left(\frac{1}{\int_\Omega w d\mu} \int_\Omega w g d\mu - x \right) \\ &= \frac{1}{\int_\Omega w d\mu} \int_\Omega w \cdot \left[(g - x) \int_0^1 \left(\Phi' \left((1-s)\, x + s g \right) - \lambda \right) ds \right] d\mu \end{aligned} \tag{2.9}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$, provided $\Phi \circ g$, $g \in L_w(\Omega, \mu)$ where

$$L_w(\Omega, \mu) := \left\{ g \middle| \int_\Omega w\, |g|\, d\mu < \infty \right\}.$$

The other equalities have similar weighted versions. However, the details are omitted.

2.2. *h*-convex functions. We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([61]). We say that $\Phi : I \to \mathbb{R}$ is a Godunova–Levin function or that Φ belongs to the class $Q(I)$ if Φ is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have

$$\Phi(tx + (1-t)y) \leq \frac{1}{t}\Phi(x) + \frac{1}{1-t}\Phi(y). \tag{2.10}$$

Some further properties of this class of functions can be found in [50], [51], [53], [79], [83] and [85]. Among others, its has been noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

The above concept can be extended for functions $\Phi : C \subseteq X \to [0,\infty)$ where C is a convex subset of the real or complex linear space X and the inequality (2.10) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $\Phi : C \subseteq X \to \mathbb{R}$ is nonnegative and convex, then it is of Godunova–Levin type.

Definition 2 ([53]). We say that a function $\Phi : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have

$$\Phi(tx + (1-t)y) \leq \Phi(x) + \Phi(y). \tag{2.11}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contains all nonnegative monotone, convex and *quasi-convex functions,* i.e. functions satisfying

$$\Phi(tx + (1-t)y) \leq \max\{\Phi(x), \Phi(y)\} \tag{2.12}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [53] and [81] while for quasi-convex functions, the reader can consult [52].

If $\Phi : C \subseteq X \to [0,\infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2.11) (or (2.12)) holds true for $x, y \in C$ and $t \in [0,1]$.

Definition 3 ([10]). Let s be a real number, $s \in (0,1]$. A function $\Phi : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$\Phi(tx + (1-t)y) \leq t^s\Phi(x) + (1-t)^s\Phi(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [2], [3], [10], [11], [47], [48], [63], [73] and [91].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and Φ are real nonnegative functions defined in J and I, respectively.

Definition 4 ([101]). Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $\Phi : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have

$$\Phi(tx + (1-t)y) \le h(t)\Phi(x) + h(1-t)\Phi(y) \tag{2.13}$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [101], [9], [76], [90], [89] and [99].

We can introduce now another class of functions.

Definition 5. We say that the function $\Phi : I \to [0, \infty) \to [0, \infty)$ is of s-Godunova–Levin type, with $s \in [0, 1]$, if

$$\Phi(tx + (1-t)y) \le \frac{1}{t^s}\Phi(x) + \frac{1}{(1-t)^s}\Phi(y), \tag{2.14}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova–Levin functions. If we denote by $Q_s(C)$ the class of s-Godunova–Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \le s_1 \le s_2 \le 1$.

For different inequalities related to these classes of functions, see [2]–[5], [9], [13]–[59], [72]–[76] and [81]–[99].

3. Inequalities for $|\Phi'|$ being h-convex, quasi-convex or log-convex.

We use the notations

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int_\Omega |k(t)|^p \, d\mu(t)\right)^{1/p} < \infty, \\ \text{if } p \ge 1, \ k \in L_p(\Omega, \mu); \\ \\ \operatorname{ess\,sup}_{t \in \Omega} |k(t)| < \infty, \\ \text{if } p = \infty, \ k \in L_\infty(\Omega, \mu) \end{cases}$$

and

$$\|\Phi\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |\Phi(s)|^p ds\right)^{1/p} < \infty, \\ \text{if } p \geq 1,\ \Phi \in L_p(0,1); \\ \\ \operatorname{ess\,sup}_{s\in[0,1]} |\Phi(s)| < \infty, \\ \text{if } p = \infty,\ \Phi \in L_\infty(0,1). \end{cases}$$

The following result holds.

Theorem 5. *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is h-convex on the interval $[a,b] \subset \mathring{I}$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\ g \in L(\Omega,\mu)$, then we have the inequality*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right| \tag{3.1}$$
$$\leq \int_0^1 h(s)\, ds \begin{cases} \|g-x\|_{\Omega,\infty}\left[|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,1}\right], \\ \text{if } \Phi' \circ g \in L(\Omega,\mu); \\ \\ \|g-x\|_{\Omega,p}\, \||\Phi'(x)| + |\Phi' \circ g|\|_{\Omega,q}, \\ \text{if } \Phi' \circ g \in L_q(\Omega,\mu),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|g-x\|_{\Omega,1}\left[|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,\infty}\right], \\ \text{if } \Phi' \circ g \in L_\infty(\Omega,\mu) \end{cases}$$

for any $x \in [a,b]$.

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \tag{3.2}$$
$$\leq \int_0^1 h(s)\, ds \begin{cases} \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \|\Phi' \circ g\|_{\Omega,1}\right], \\ \text{if } \Phi' \circ g \in L(\Omega,\mu); \\ \\ \left\|g - \int_\Omega g d\mu\right\|_{\Omega,p} \left\|\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + |\Phi' \circ g|\right\|_{\Omega,q}, \\ \text{if } \Phi' \circ g \in L_q(\Omega,\mu),\ p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|g - \int_\Omega g d\mu\right\|_{\Omega,1}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \|\Phi' \circ g\|_{\Omega,\infty}\right], \\ \text{if } \Phi' \circ g \in L_\infty(\Omega,\mu) \end{cases}$$

and

$$
(3.3)\quad \begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\
&\le \int_0^1 h(s)\,ds \begin{cases}
\left\|g-\frac{a+b}{2}\right\|_{\Omega,\infty}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,1}\right], \\
\text{if } \Phi'\circ g\in L(\Omega,\mu); \\
\\
\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left\|\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left|\Phi'\circ g\right|\right\|_{\Omega,q}, \\
\text{if } \Phi'\circ g\in L_q(\Omega,\mu),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left\|g-\frac{a+b}{2}\right\|_{\Omega,1}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right], \\
\text{if } \Phi'\circ g\in L_\infty(\Omega,\mu)
\end{cases} \\
&\le \frac{1}{2}(b-a)\int_0^1 h(s)\,ds \begin{cases}
\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,1}\right]; \\
\\
\left\|\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left|\Phi'\circ g\right|\right\|_{\Omega,q}, \\
\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right].
\end{cases}
\end{aligned}
$$

Proof. We have from (2.2) that

$$
(3.4)\quad \left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| \le \int_\Omega |g-x|\left|\int_0^1 \Phi'((1-s)x+sg)\,ds\right| d\mu,
$$

for any $x\in[a,b]$.

Utilising Hölder's inequality for the μ-measurable functions $F, G:\Omega\to\mathbb{C}$,

$$
\left|\int_\Omega FG d\mu\right| \le \left(\int_\Omega |F|^p d\mu\right)^{1/p}\left(\int_\Omega |G|^q d\mu\right)^{1/q},
$$

$p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and

$$
\left|\int_\Omega FG d\mu\right| \le \operatorname*{ess\,sup}_{t\in\Omega}|F(t)|\int_\Omega |G|\,d\mu,
$$

we have

$$B := \int_\Omega |g-x| \left| \int_0^1 \Phi'((1-s)x+sg)\,ds \right| d\mu$$

$$\tag{3.5} \leq \begin{cases} \operatorname*{ess\,sup}_{t\in\Omega} |g(t)-x| \int_\Omega \left| \int_0^1 \Phi'((1-s)x+sg)\,ds \right| d\mu; \\ \\ \left(\int_\Omega |g-x|^p\,d\mu\right)^{1/p} \left(\int_\Omega \left| \int_0^1 \Phi'((1-s)x+sg)\,ds \right|^q d\mu\right)^{1/q}, \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\Omega |g-x|\,d\mu \ \operatorname*{ess\,sup}_{t\in\Omega} \left| \int_0^1 \Phi'((1-s)x+sg)\,ds \right|, \end{cases}$$

for any $x\in[a,b]$.

Since $|\Phi'|$ is h-convex on the interval $[a,b]$, then we have for any $t\in\Omega$ that

$$\begin{aligned} \left| \int_0^1 \Phi'((1-s)x+sg(t))\,ds \right| &\leq \int_0^1 \left|\Phi'((1-s)x+sg(t))\right| ds \\ &\leq \left|\Phi'(x)\right| \int_0^1 h(1-s)\,ds + \left|\Phi'(g(t))\right| \int_0^1 h(s)\,ds \\ &= \left[\left|\Phi'(x)\right| + \left|\Phi'(g(t))\right|\right] \int_0^1 h(s)\,ds, \end{aligned}$$

for any $x\in[a,b]$.

This implies that

$$\tag{3.6} \begin{aligned} &\int_\Omega \left| \int_0^1 \Phi'((1-s)x+sg)\,ds \right| d\mu \\ &\qquad \leq \int_0^1 h(s)\,ds \left[\left|\Phi'(x)\right| + \int_\Omega \left|\Phi'\circ g\right| d\mu\right] \end{aligned}$$

for any $x\in[a,b]$.

We have for any $t\in\Omega$ that

$$\begin{aligned} \left| \int_0^1 \Phi'((1-s)x+sg(t))\,ds \right|^q &\leq \left[\int_0^1 \left|\Phi'((1-s)x+sg(t))\right| ds\right]^q \\ &\leq \left[\left[\left|\Phi'(x)\right| + \left|\Phi'(g(t))\right|\right] \int_0^1 h(s)\,ds\right]^q \\ &= \left[\int_0^1 h(s)\,ds\right]^q \left[\left|\Phi'(x)\right| + \left|\Phi'(g(t))\right|\right]^q \end{aligned}$$

for any $x\in[a,b]$.

This implies

$$\left(\int_{\Omega}\left|\int_0^1 \Phi'\left((1-s)x+sg\right)ds\right|^q d\mu\right)^{1/q}$$

$$\leq \int_0^1 h(s)\,ds\left[\int_{\Omega}\left[\left|\Phi'(x)\right|+\left|\Phi'(g(t))\right|\right]^q d\mu\right]^{1/q} \tag{3.7}$$

$$=\int_0^1 h(s)\,ds\left[\int_{\Omega}\left[\left|\Phi'(x)\right|+\left|\Phi'\circ g\right|\right]^q d\mu\right]^{1/q}.$$

Also

$$\operatorname*{ess\,sup}_{t\in\Omega}\left|\int_0^1 \Phi'\left((1-s)x+sg\right)ds\right|$$

$$\leq\left[\left|\Phi'(x)\right|+\operatorname*{ess\,sup}_{t\in\Omega}\left|\Phi'(g(t))\right|\right]\int_0^1 h(s)\,ds \tag{3.8}$$

$$=\left[\left|\Phi'(x)\right|+\operatorname*{ess\,sup}_{t\in\Omega}\left|\Phi'\circ g\right|\right]\int_0^1 h(s)\,ds$$

for any $x\in[a,b]$.

Making use of (3.6)–(3.8), we get the desired result (3.1). □

Remark 5. With the assumptions of Theorem 5 and if $|\Phi'|$ is convex on the interval $[a,b]$, then $\int_0^1 h(s)\,ds=\frac{1}{2}$ and the inequalities (3.1)–(3.3) hold with $\frac{1}{2}$ instead of $\int_0^1 h(s)\,ds$. If $|\Phi'|$ is of s-Godunova–Levin type, with $s\in[0,1)$ on the interval $[a,b]$, then $\int_0^1 \frac{1}{t^s}dt=\frac{1}{1-s}$ and the inequalities (3.1)–(3.3) hold with $\frac{1}{1-s}$ instead of $\int_0^1 h(s)\,ds$.

Following [52], we say that for an interval $I\subseteq\mathbb{R}$, the mapping $h:I\to\mathbb{R}$ is *quasi-monotone* on I if it is either monotone on $I=[c,d]$ or monotone nonincreasing on a proper subinterval $[c,c']\subset I$ and monotone nondecreasing on $[c',d]$.

The class $QM(I)$ of quasi-monotone functions on I provides an immediate characterization of quasi-convex functions [52].

Proposition 2. *Suppose $I\subseteq\mathbb{R}$. Then the following statements are equivalent for a function $h:I\to\mathbb{R}$:*

(a) *$h\in QM(I)$;*
(b) *on any subinterval of I, h achieves its supremum at an end point;*
(c) *h is quasi-convex.*

As examples of quasi-convex functions we may consider the class of monotonic functions on an interval I for the class of convex functions on that interval.

Theorem 6. *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is quasi-convex on the interval $[a,b] \subset \mathring{I}$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega,\mu)$ and $\Phi' \circ g \in L_\infty(\Omega,\mu)$, then we have the inequality*

$$\begin{aligned} \left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right| &\le \int_\Omega |g-x| \max\left\{\left|\Phi'(x)\right|, \left|\Phi' \circ g\right|\right\} d\mu \\ &\le \max\left\{\left|\Phi'(x)\right|, \left\|\Phi' \circ g\right\|_{\Omega,\infty}\right\} \|g-x\|_{\Omega,1} \end{aligned} \tag{3.9}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\ &\le \int_\Omega \left|g - \int_\Omega g d\mu\right| \max\left\{\left|\Phi'\left(\int_\Omega g d\mu\right)\right|, \left|\Phi' \circ g\right|\right\} d\mu \\ &\le \max\left\{\left|\Phi'(x)\right|, \left\|\Phi' \circ g\right\|_{\Omega,\infty}\right\} \left\|g - \int_\Omega g d\mu\right\|_{\Omega,1} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\ &\le \int_\Omega \left|g - \frac{a+b}{2}\right| \max\left\{\left|\Phi'\left(\frac{a+b}{2}\right)\right|, \left|\Phi' \circ g\right|\right\} d\mu \\ &\le \max\left\{\left|\Phi'\left(\frac{a+b}{2}\right)\right|, \left\|\Phi' \circ g\right\|_{\Omega,\infty}\right\} \left\|g - \frac{a+b}{2}\right\|_{\Omega,1}. \end{aligned} \tag{3.11}$$

Proof. From (3.4) we have

$$\begin{aligned} \left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right| &\le \int_\Omega |g-x| \left(\int_0^1 \left|\Phi'((1-s)x + sg)\right| ds\right) d\mu \\ &\le \int_\Omega |g-x| \max\left\{\left|\Phi'(x)\right|, \left|\Phi' \circ g\right|\right\} d\mu, \end{aligned} \tag{3.12}$$

for any $x \in [a,b]$.

Observe that

$$\left|\left(\Phi' \circ g\right)(t)\right| \le \left\|\Phi' \circ g\right\|_{\Omega,\infty} \text{ for almost every } t \in \Omega$$

and then

$$\begin{aligned}
&\int_\Omega |g-x| \max\left\{\left|\Phi'(x)\right|, \left|\Phi'\circ g\right|\right\} d\mu \\
(3.13)\qquad &\le \int_\Omega |g-x| \max\left\{\left|\Phi'(x)\right|, \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right\} d\mu \\
&= \max\left\{\left|\Phi'(x)\right|, \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right\} \int_\Omega |g-x|\, d\mu,
\end{aligned}$$

for any $x\in[a,b]$.

Using (3.12) and (3.13), we get the desired result (3.9). □

In what follows, I will denote an interval of real numbers. A function $f: I\to(0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for any $x,y\in I$ and $t\in[0,1]$ one has the inequality

$$f(tx+(1-t)y) \le [f(x)]^t [f(y)]^{1-t}. \tag{3.14}$$

We note that if f and g are convex and g is increasing, then $g\circ f$ is convex, moreover, since $f=\exp[\log f]$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (3.14) since, by the arithmetic-geometric mean inequality we have

$$[f(x)]^t [f(y)]^{1-t} \le tf(x)+(1-t)f(y) \tag{3.15}$$

for all $x,y\in I$ and $t\in[0,1]$.

Theorem 7. *Let $\Phi: I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is log-convex on the interval $[a,b]\subset\mathring{I}$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $\Phi'\circ g$, $g\in L(\Omega,\mu)$ then we have the inequality*

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| \\
&\le \int_\Omega |g-x|\, L\left(\left|\Phi'\circ g\right|, \left|\Phi'(x)\right|\right) d\mu \\
(3.16)\qquad &\le \frac{1}{2}\left[\left|\Phi'(x)\right|\int_\Omega |g-x|\,d\mu + \int_\Omega |g-x|\left|\Phi'\circ g\right| d\mu\right] \\
&\left(\le \frac{1}{2}\left[\left|\Phi'(x)\right| + \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right]\|g-x\|_{\Omega,1} \text{ if } \Phi'\circ g\in L_\infty(\Omega,\mu)\right)
\end{aligned}$$

for any $x\in[a,b]$, where $L(\cdot,\cdot)$ is the logarithmic mean, namely for $\alpha,\beta>0$

$$L(\alpha,\beta) := \begin{cases} \dfrac{\alpha-\beta}{\ln\alpha-\ln\beta}, & \alpha\ne\beta, \\ \alpha, & \alpha=\beta. \end{cases}$$

In particular, we have

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\
&\leq \int_\Omega \left|g - \int_\Omega g d\mu\right| L\left(\left|\Phi'\circ g\right|, \left|\Phi'\left(\int_\Omega g d\mu\right)\right|\right) d\mu \\
&\leq \frac{1}{2}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| \int_\Omega \left|g - \int_\Omega g d\mu\right| d\mu + \int_\Omega \left|g - \int_\Omega g d\mu\right| \left|\Phi'\circ g\right| d\mu\right] \quad (3.17)\\
&\left(\leq \frac{1}{2}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right] \left\|g - \int_\Omega g d\mu\right\|_{\Omega,1}\right. \\
&\qquad \left.\textit{if } \Phi'\circ g \in L_\infty(\Omega,\mu)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\
&\leq \int_\Omega \left|g - \frac{a+b}{2}\right| L\left(\left|\Phi'\circ g\right|, \left|\Phi'\left(\frac{a+b}{2}\right)\right|\right) d\mu \\
&\leq \frac{1}{2}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu + \int_\Omega \left|g - \frac{a+b}{2}\right| \left|\Phi'\circ g\right| d\mu\right] \quad (3.18)\\
&\left(\leq \frac{1}{2}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right] \left\|g - \frac{a+b}{2}\right\|_{\Omega,1}\right. \\
&\qquad \left.\textit{if } \Phi'\circ g \in L_\infty(\Omega,\mu)\right).
\end{aligned}$$

Proof. From (3.4) we have

$$\begin{aligned}
\left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| &\leq \int_\Omega |g-x| \left(\int_0^1 \left|\Phi'((1-s)x + sg)\right| ds\right) d\mu \\
&\leq \int_\Omega |g-x| \left(\int_0^1 \left|\Phi'(x)\right|^{1-s} \left|\Phi'\circ g\right|^s ds\right) d\mu, \quad (3.19)
\end{aligned}$$

for any $x \in [a,b]$.

Since, for any $C > 0$, one has

$$\int_0^1 C^\lambda d\lambda = \frac{C-1}{\ln C},$$

then for any $t \in \Omega$ we have

$$\begin{aligned}
&\int_0^1 \left|\Phi'(x)\right|^{1-s} \left|\Phi'(g(t))\right|^s ds = \left|\Phi'(x)\right| \int_0^1 \left|\frac{\Phi'(g(t))}{\Phi'(x)}\right|^s ds \\
&= \left|\Phi'(x)\right| \frac{\left|\frac{\Phi'(g(t))}{\Phi'(x)}\right| - 1}{\ln\left|\frac{\Phi'(g(t))}{\Phi'(x)}\right|} \\
&= \frac{|\Phi'(g(t))| - |\Phi'(x)|}{\ln|\Phi'(g(t))| - \ln|\Phi'(x)|} \\
&= L\left(\left|\Phi'(g(t))\right|, \left|\Phi'(x)\right|\right),
\end{aligned} \tag{3.20}$$

for any $x \in [a, b]$.

Making use of (3.19) and (3.20), we get the first inequality in (3.16).

The second inequality in (3.16) follows by the fact that

$$L(\alpha, \beta) \le \frac{\alpha + \beta}{2} \text{ for any } \alpha, \beta > 0.$$

The last inequality in (3.16) is obvious. □

4. Inequalities for $|\Phi'|^q$ being h-convex or log-convex.

We have:

Theorem 8. *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that for $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|\Phi'|^q$ is h-convex on the interval $[a, b] \subset \mathring{I}$.*

If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$ and $\Phi' \circ g \in L_q(\Omega, \mu)$, then we have the inequality

$$\begin{aligned}
&\left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right| \\
&\le \left(\int_0^1 h(s) ds\right)^{1/q} \|g - x\|_{\Omega,p} \left(\left|\Phi'(x)\right|^q + \int_\Omega \left|\Phi' \circ g\right|^q d\mu\right)^{1/q} \\
&\le \left(\int_0^1 h(s) ds\right)^{1/q} \|g - x\|_{\Omega,p} \left(\left|\Phi'(x)\right| + \left\|\Phi' \circ g\right\|_{\Omega,q}\right)
\end{aligned} \tag{4.1}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\
&\quad\le \left(\int_0^1 h(s)\,ds\right)^{1/q} \\
&\qquad\times \left\|g-\int_\Omega g d\mu\right\|_{\Omega,p}\left(\left|\Phi'\left(\int_\Omega g d\mu\right)\right|^q + \int_\Omega \left|\Phi'\circ g\right|^q d\mu\right)^{1/q} \\
&\quad\le \left(\int_0^1 h(s)\,ds\right)^{1/q} \\
&\qquad\times \left\|g-\int_\Omega g d\mu\right\|_{\Omega,p}\left(\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,q}\right)
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \left(\int_0^1 h(s)\,ds\right)^{1/q} \\
&\qquad\times \left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\left|\Phi'\left(\frac{a+b}{2}\right)\right|^q + \int_\Omega \left|\Phi'\circ g\right|^q d\mu\right)^{1/q} \\
&\quad\le \left(\int_0^1 h(s)\,ds\right)^{1/q} \\
&\qquad\times \left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,q}\right).
\end{aligned} \tag{4.3}$$

Proof. From the proof of Theorem 5 we have

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| \\
&\le \int_\Omega |g-x|\left|\int_0^1 \Phi'((1-s)x+sg)\,ds\right| d\mu \\
&\le \left(\int_\Omega |g-x|^p d\mu\right)^{1/p}\left(\int_\Omega \left|\int_0^1 \Phi'((1-s)x+sg)\,ds\right|^q d\mu\right)^{1/q} \\
&\le \left(\int_\Omega |g-x|^p d\mu\right)^{1/p}\left(\int_\Omega \left(\int_0^1 \left|\Phi'((1-s)x+sg)\right|^q ds\right) d\mu\right)^{1/q}
\end{aligned} \tag{4.4}$$

for $p>1$, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $x\in[a,b]$.

Since $|\Phi'|^q$ is h-convex on the interval $[a,b]$, then

$$\begin{aligned}
&\int_0^1 \left|\Phi'\left((1-s)x+sg(t)\right)\right|^q ds \\
&\leq \left|\Phi'(x)\right|^q \int_0^1 h(1-s)\,ds + \left|\Phi'(g(t))\right|^q \int_0^1 h(s)\,ds \\
&= \left[\left|\Phi'(x)\right|^q + \left|\Phi'(g(t))\right|^q\right] \int_0^1 h(s)\,ds
\end{aligned}$$

for any $x \in [a,b]$ and $t \in \Omega$.

Therefore

$$\begin{aligned}
&\left(\int_\Omega \left(\int_0^1 \left|\Phi'\left((1-s)x+sg\right)\right|^q ds\right) d\mu\right)^{1/q} \\
&\leq \left(\int_\Omega \left(\left[\left|\Phi'(x)\right|^q + \left|\Phi'(g(t))\right|^q\right] \int_0^1 h(s)\,ds\right) d\mu\right)^{1/q} \\
&= \left(\int_0^1 h(s)\,ds\right)^{1/q} \left(\left|\Phi'(x)\right|^q + \int_\Omega \left|\Phi' \circ g\right|^q d\mu\right)^{1/q}
\end{aligned} \tag{4.5}$$

for any $x \in [a,b]$.

This proves the first inequality in (4.1).

Now, we observe that the following elementary inequality holds:

$$(\alpha+\beta)^r \geq (\leq)\, \alpha^r + \beta^r \tag{4.6}$$

for any $\alpha, \beta \geq 0$ and $r \geq 1$ $(0 < r < 1)$.

Indeed, if we consider the function $f_r : [0,\infty) \to \mathbb{R}$, $f_r(t) = (t+1)^r - t^r$ we have $f_r'(t) = r\left[(t+1)^{r-1} - t^{r-1}\right]$. Observe that for $r > 1$ and $t > 0$ we have that $f_r'(t) > 0$ showing that f_r is strictly increasing on the interval $[0,\infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0,\ \alpha \geq 0)$ we have $f_r(t) > f_r(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^r - \left(\frac{\alpha}{\beta}\right)^r > 1$, i.e., the desired inequality (4.6).

For $r \in (0,1)$ we have that f_r is strictly decreasing on $[0,\infty)$ which proves the second case in (4.6).

Making use of (4.6) for $r = 1/q \in (0,1)$, we have

$$\left(\left|\Phi'(x)\right|^q + \int_\Omega \left|\Phi' \circ g\right|^q d\mu\right)^{1/q} \leq \left|\Phi'(x)\right| + \left(\int_\Omega \left|\Phi' \circ g\right|^q d\mu\right)^{1/q}$$

and then we get the second part of (4.1). □

Finally, we have:

Theorem 9. *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that for $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|\Phi'|^q$ is log-convex on the interval $[a, b] \subset \mathring{I}$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$ and $\Phi' \circ g \in L_q(\Omega, \mu)$, then we have the inequality*

$$\begin{aligned}
&\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) \right| \\
&\leq \|g - x\|_{\Omega,p} \left(\int_\Omega L\left(|\Phi' \circ g|^q, |\Phi'(x)|^q \right) d\mu \right)^{1/q} \\
&\leq \frac{1}{2^{1/q}} \|g - x\|_{\Omega,p} \left[|\Phi'(x)|^q + \int_\Omega |\Phi' \circ g|^q d\mu \right]^{1/q} \\
&\leq \frac{1}{2^{1/q}} \|g - x\|_{\Omega,p} \left[|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,q} \right]
\end{aligned} \tag{4.7}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned}
&\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu \right) \right| \\
&\leq \left\| g - \int_\Omega g d\mu \right\|_{\Omega,p} \left(\int_\Omega L\left(|\Phi' \circ g|^q, \left| \Phi'\left(\int_\Omega g d\mu \right) \right|^q \right) d\mu \right)^{1/q} \\
&\leq \frac{1}{2^{1/q}} \left\| g - \int_\Omega g d\mu \right\|_{\Omega,p} \left[\left| \Phi'\left(\int_\Omega g d\mu \right) \right|^q + \int_\Omega |\Phi' \circ g|^q d\mu \right]^{1/q} \\
&\leq \frac{1}{2^{1/q}} \left\| g - \int_\Omega g d\mu \right\|_{\Omega,p} \left[\left| \Phi'\left(\int_\Omega g d\mu \right) \right| + \|\Phi' \circ g\|_{\Omega,q} \right]
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
&\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2} \right) \right| \\
&\leq \left\| g - \frac{a+b}{2} \right\|_{\Omega,p} \left(\int_\Omega L\left(|\Phi' \circ g|^q, \left| \Phi'\left(\frac{a+b}{2} \right) \right|^q \right) d\mu \right)^{1/q} \\
&\leq \frac{1}{2^{1/q}} \left\| g - \frac{a+b}{2} \right\|_{\Omega,p} \left[\left| \Phi'\left(\frac{a+b}{2} \right) \right|^q + \int_\Omega |\Phi' \circ g|^q d\mu \right]^{1/q} \\
&\leq \frac{1}{2^{1/q}} \left\| g - \frac{a+b}{2} \right\|_{\Omega,p} \left[\left| \Phi'\left(\frac{a+b}{2} \right) \right| + \|\Phi' \circ g\|_{\Omega,q} \right].
\end{aligned} \tag{4.9}$$

Proof. Since $|\Phi'|^q$ is log-convex on the interval $[a,b]$, then

$$\begin{aligned}\int_0^1 \left|\Phi'((1-s)x+sg(t))\right|^q ds &\le \int_0^1 \left|\Phi'(x)\right|^{q(1-s)} |g(t)|^{sq}\, ds\\ &= \left|\Phi'(x)\right|^q \int_0^1 \left|\frac{g(t)}{\Phi'(x)}\right|^{sq} ds\\ &= L\left(\left|\Phi'(g(t))\right|^q, \left|\Phi'(x)\right|^q\right)\end{aligned}$$

for any $x \in [a,b]$ and $t \in \Omega$.

Then

$$\begin{aligned}&\left(\int_\Omega \left(\int_0^1 \left|\Phi'((1-s)x+sg)\right|^q ds\right) d\mu\right)^{1/q}\\ &\qquad\le \left(\int_\Omega L\left(\left|\Phi'\circ g\right|^q, \left|\Phi'(x)\right|^q\right) d\mu\right)^{1/q}\end{aligned}$$

and by (4.4) we get the first inequality in (4.7).

Since, in general

$$L(\alpha,\beta) \le \frac{\alpha+\beta}{2} \text{ for any } \alpha,\beta>0,$$

then

$$\begin{aligned}\int_\Omega L\left(\left|\Phi'\circ g\right|^q, \left|\Phi'(x)\right|^q\right) d\mu &\le \frac{1}{2}\int_\Omega \left[\left|\Phi'\circ g\right|^q + \left|\Phi'(x)\right|^q\right] d\mu\\ &= \frac{1}{2}\left[\left|\Phi'(x)\right|^q + \int_\Omega \left|\Phi'\circ g\right|^q d\mu\right]\end{aligned}$$

and we get the second inequality in (4.7).

The last part is obvious. □

5. Applications for *f*-divergence. One of the important issues in many applications of probability theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [67], Kullback and Leibler [74], Rényi [87], Havrda and Charvat [65], Kapur [70], Sharma and Mittal [92], Burbea and Rao [12], Rao [86], Lin [75], Csiszár [20], Ali and Silvey [1], Vajda [100], Shioya and Da-Te [94] and others (see for example [77] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [86], genetics [77], finance, economics, and political science [93], [96], [97], biology [84], the analysis of contingency tables [62], approximation of probability distributions [18], [71], signal processing [68], [69] and pattern recognition [7], [17]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be

$$\mathcal{P} := \left\{ p \mid p : \Omega \to \mathbb{R},\ p(t) \geq 0,\ \int_\Omega p(t)\, d\mu(t) = 1 \right\}.$$

The Kullback–Leibler divergence [74] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t) \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.1}$$

where ln is to base e.

In information theory and statistics, various divergences are applied in addition to the Kullback–Leibler divergence. These are: *variation distance* D_v, *Hellinger distance* D_H [66], χ^2*-divergence* D_{χ^2}, α*-divergence* D_α, *Bhattacharyya distance* D_B [8], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [67], *triangular discrimination* D_Δ [98], etc... They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(t) - q(t)|\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.2}$$

$$D_H(p,q) := \int_\Omega \left| \sqrt{p(t)} - \sqrt{q(t)} \right| d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.3}$$

$$D_{\chi^u}(p,q) := \int_\Omega p(t) \left[\left(\frac{q(t)}{p(t)} \right)^r - 1 \right] d\mu(t),\ u \geq 2,\ p,q \in \mathcal{P}; \tag{5.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2} \left[1 - \int_\Omega [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t) \right], \quad p,q \in \mathcal{P}; \tag{5.5}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(t)\, q(t)}\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.6}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(t)\, q(t)}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.7}$$

$$D_J(p,q) := \int_\Omega [p(t) - q(t)] \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.8}$$

$$D_\Delta(p,q) := \int_\Omega \frac{[p(t) - q(t)]^2}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}. \tag{5.9}$$

For other divergence measures, see the paper [70] by Kapur or the online book [95] by Taneja.

Csiszár f-divergence is defined as follows [21]:

$$I_f(p,q) := \int_\Omega p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)–(5.9), are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example [95]). For the basic properties of Csiszár f-divergence see [21], [22] and [100].

The following result holds:

Proposition 3. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Assume that $p,q\in\mathcal{P}$ and there exist constants $0<r<1<R<\infty$ such that*

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t\in\Omega. \tag{5.11}$$

If $|f'|$ is h-convex on the interval $[r,R]$, then we have the inequalities

$$0 \le I_f(p,q) \le \int_0^1 h(s)\,ds \begin{cases} (R-r)\left[|\Phi'(1)| + I_{|f'|}(p,q)\right], \\ D_v(p,q)\left[|\Phi'(1)| + \|f'\|_{[r,R],\infty}\right]. \end{cases} \tag{5.12}$$

Proof. Applying the inequality (3.2), we have

$$\begin{aligned} &\left|\int_\Omega p(t) f\left(\frac{q(t)}{p(t)}\right) d\mu(t) - f(1)\right| \\ &\le \int_0^1 h(s)\,ds \\ &\quad\times \begin{cases} \operatorname{ess\,sup}_{t\in\Omega}\left|\frac{q(t)}{p(t)} - 1\right| \left[|\Phi'(1)| + \int_\Omega p(t)\left|f'\left(\frac{q(t)}{p(t)}\right)\right| d\mu(t)\right], \\ \|q-p\|_{\Omega,1}\left[|\Phi'(1)| + \operatorname{ess\,sup}_{t\in\Omega}\left|f'\left(\frac{q(t)}{p(t)}\right)\right|\right] \end{cases} \\ &\le \int_0^1 h(s)\,ds \\ &\quad\times \begin{cases} (R-r)\left[|\Phi'(1)| + I_{|f'|}(p,q)\right], \\ D_v(p,q)\left[|\Phi'(1)| + \operatorname{ess\,sup}_{x\in[r,R]}|f'(x)|\right] \end{cases} \end{aligned}$$

and the inequality (5.12) is obtained. □

Consider the convex function $f(x)=x^u-1$, $u\ge 2$. Then $f(1)=0$, $f'(x)=ux^{u-1}$ and $|f'|$ is convex on the interval $[r,R]$ for any $0<r<1<R<\infty$.

Then by (5.12) we have

$$0 \le D_{\chi^u}(p,q) \le \frac{1}{2}u \begin{cases} (R-r)\left[1+D_{\chi^{u-1}}(p,q)\right], \\ \\ D_v(p,q)\left(1+R^{u-1}\right), \end{cases} \tag{5.13}$$

provided

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

If we consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=-\ln t$, then

$$\begin{aligned} I_f(p,q) &:= -\int_\Omega p(t)\ln\left[\frac{q(t)}{p(t)}\right]d\mu(t) = \int_\Omega p(t)\ln\left[\frac{p(t)}{q(t)}\right]d\mu(t) \\ &= D_{KL}(p,q). \end{aligned}$$

We have $f'(t)=-\frac{1}{t}$ and $|f'|$ is convex on the interval $[r,R]$ for any $0<r<1<R<\infty$. If we apply the inequality (5.12) we have

$$0 \le D_{KL}(p,q) \le \frac{1}{2} \begin{cases} (R-r)\left[2+D_{\chi^2}(q,p)\right], \\ \\ \frac{r+1}{r}D_v(p,q), \end{cases} \tag{5.14}$$

provided

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

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