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PAWEŁ BEDNARZ, ANNA KOSIOROWSKA
and ADRIAN MICHALSKI

Special bihyperbolic numbers and their connections with triangular tables and matrices

Abstract. In this paper we express special bihyperbolic numbers as paradeterminants and parapermanents of some triangular matrices. Moreover, by applying the connections between these parameters of triangular tables and the determinants and permanents of lower Hessenberg matrices, we obtain another expressions of these numbers, using matrices which are not triangular.

1. Introduction. Let $n \geq 0$ be an integer. The nth balancing number B_n, Lucas-balancing number C_n, Mersenne number M_n and Mersenne–Lucas number H_n are given by the following recursive definitions:

$$\begin{aligned} B_n &= 6B_{n-1} - B_{n-2}, \text{ for } n \geq 2 \text{ with } B_0 = 0,\ B_1 = 1, \\ C_n &= 6C_{n-1} - C_{n-2}, \text{ for } n \geq 2 \text{ with } C_0 = 1,\ C_1 = 3, \\ M_n &= 3M_{n-1} - 2M_{n-2}, \text{ for } n \geq 2 \text{ with } M_0 = 0,\ M_1 = 1, \\ H_n &= 3H_{n-1} - 2H_{n-2}, \text{ for } n \geq 2 \text{ with } H_0 = 2,\ H_1 = 3. \end{aligned}$$

In Table 1 we list first numbers of sequences defined above.

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n	0	1	2	3	4	5	5	6
B_n	0	1	6	35	204	1189	6930	40391
C_n	1	3	17	99	577	3363	19601	114243
M_n	0	1	3	7	15	31	63	127
H_n	2	3	5	9	17	33	65	129

TABLE 1. The first terms of sequences B_n, C_n, M_n, H_n.

Balancing numbers were introduced by Behera and Panda in [2]. Later, in [18], Panda defined Lucas-balancing numbers. These two sequences were extensively studied, also by considering some generalizations. For more details, see for example [8, 11, 20]. The literature on Mersenne and Mersenne–Lucas sequences is also broad, see [4, 9, 16, 22] among others.

For the integer sequences considered in this paper: balancing numbers, Lucas-balancing numbers, Mersenne numbers and Mersenne–Lucas numbers, we also provide their corresponding entries in the OEIS (The On-Line Encyclopedia of Integer Sequences):

B_n : A001109, C_n : A001541, M_n : A000225, H_n : A000051.

The hyperbolic unit j was introduced in 1848 by J. Cockle in [10]. Elements of the set $\mathbb{H} = \{a + bj : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}$ are known as *hyperbolic numbers*. The hyperbolic numbers, which are known also as *split complex numbers* or *double numbers* were studied for the first time in the 19th century. Originally, the non-Euclidean framework was described by Yaglom [23], while a systematic modern treatment of hypercomplex systems was given by Olariu [17] and by Kantor and Solodovnikov [15]. Fundamental algebraic properties of hyperbolic numbers were studied in detail by Rochon and Shapiro [21].

One of the generalizations of hyperbolic numbers was introduced in [19]. Let us denote by $\mathbb{H}_2$ the set of all numbers of the form

$$\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and operators j_1, j_2, $j_3 \notin \mathbb{R}$ satisfy conditions

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1j_2 = j_2j_1 = j_3,\ j_1j_3 = j_3j_1 = j_2,\ j_2j_3 = j_3j_2 = j_1.$$

The elements of the set $\mathbb{H}_2$ are called *bihyperbolic numbers*.

Addition and multiplication of bihyperbolic numbers are performed analogously to algebraic expressions. These operations are associative and commutative on $\mathbb{H}_2$. Moreover, multiplication is distributive over addition, so $(\mathbb{H}_2, +, \cdot)$ is a commutative ring.

Bihyperbolic numbers are well known in the literature, their properties can be found for example in [3, 21]. Recently, bihyperbolic extensions of classical integer sequences have also been investigated; see, for example, the bihyperbolic Tribonacci-type sequences introduced in [14].

Some special cases of bihyperbolic numbers were studied in the literature. In particular, in [7], authors defined bihyperbolic numbers of the Fibonacci type. Later, following this research, other types of bihyperbolic numbers were introduced and we will focus on four of them:

- bihyperbolic balancing numbers and bihyperbolic Lucas-balancing numbers defined in [6],
- bihyperbolic Mersenne numbers and bihyperbolic Mersenne–Lucas numbers defined in [5].

Let $n \geq 0$ be an integer. The nth bihyperbolic balancing number BhB_n, bihyperbolic Lucas-balancing number BhC_n, bihyperbolic Mersenne number BhM_n and bihyperbolic Mersenne–Lucas number BhH_n are defined in the following way:

$$BhB_n = B_n + j_1 B_{n+1} + j_2 B_{n+2} + j_3 B_{n+3}, \tag{1}$$

$$BhC_n = C_n + j_1 C_{n+1} + j_2 C_{n+2} + j_3 C_{n+3}, \tag{2}$$

$$BhM_n = M_n + j_1 M_{n+1} + j_2 M_{n+2} + j_3 M_{n+3}, \tag{3}$$

$$BhH_n = H_n + j_1 H_{n+1} + j_2 H_{n+2} + j_3 H_{n+3}, \tag{4}$$

where B_n is the nth balancing number, C_n is the nth Lucas-balancing number, M_n is the nth Mersenne number and H_n is the nth Mersenne–Lucas number.

Below we list first four terms of each sequence mentioned above.

$$\begin{aligned}
BhB_0 &= j_1 + 6j_2 + 35j_3,\\
BhB_1 &= 1 + 6j_1 + 35j_2 + 204j_3,\\
BhB_2 &= 6 + 35j_1 + 204j_2 + 1189j_3,\\
BhB_3 &= 35 + 204j_1 + 1189j_2 + 6930j_3,\\
&\vdots
\end{aligned} \tag{5}$$

$$\begin{aligned}
BhC_0 &= 1 + 3j_1 + 17j_2 + 99j_3,\\
BhC_1 &= 3 + 17j_1 + 99j_2 + 577j_3,\\
BhC_2 &= 17 + 99j_1 + 577j_2 + 3363j_3,\\
BhC_3 &= 99 + 577j_1 + 3363j_2 + 19601j_3,\\
&\vdots
\end{aligned} \tag{6}$$

$$\begin{aligned} BhM_0 &= j_1 + 3j_2 + 7j_3, \\ BhM_1 &= 1 + 3j_1 + 7j_2 + 15j_3, \\ BhM_2 &= 3 + 7j_1 + 15j_2 + 31j_3, \\ BhM_3 &= 7 + 15j_1 + 31j_2 + 63j_3, \\ &\vdots \end{aligned} \tag{7}$$

$$\begin{aligned} BhH_0 &= 2 + 3j_1 + 5j_2 + 9j_3, \\ BhH_1 &= 3 + 5j_1 + 9j_2 + 17j_3, \\ BhH_2 &= 5 + 9j_1 + 17j_2 + 33j_3, \\ BhH_3 &= 9 + 17j_1 + 33j_2 + 65j_3, \\ &\vdots \end{aligned} \tag{8}$$

The following recurrence relations concerning the numbers BhB_n, BhC_n, BhM_n, BhH_n were proved in [5, 6].

Theorem 1.1 ([6]). *Let $n \geq 2$ be an integer. Then*

(i) $BhB_n = 6BhB_{n-1} - BhB_{n-2}$,

(ii) $BhC_n = 6BhC_{n-1} - BhC_{n-2}$,

where BhB_0, BhB_1, BhC_0, BhC_1 are given by (5), (6), *respectively.*

Theorem 1.2 ([5]). *Let $n \geq 2$ be an integer. Then*

(i) $BhM_n = 3BhM_{n-1} - 2BhM_{n-2}$,

(ii) $BhH_n = 3BhH_{n-1} - 2BhH_{n-2}$,

where BhM_0, BhM_1, BhH_0, BhH_1 are given by (7), (8), *respectively.*

For details concerning numbers $BhB_n, BhC_n, BhM_n, BhH_n$, including, among others, results about generating function and Binet's formulas see [5, 6].

In [1], Bednarz and Szynal-Liana proved relations between bihyperbolic numbers of the Fibonacci type and parameters of some special types of triangular tables and matrices. Following their research, in this paper we will express bihyperbolic numbers defined by formulas (1)–(4) as paradeterminants and parapermanents of triangular matrices and as determinants and permanents of matrices. Before we do it, let us remind some facts about triangular matrices, paradeterminants and parapermanents.

2. Triangular matrices, paradeterminants and parapermanents. An array of numbers from some field K of the form

$$A_n = \begin{bmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ \vdots & \vdots & \ddots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}_{n\times n}$$

is known as a triangular matrix of order n.

It is important to acknowledge (see [24]) that a triangular matrix defined above is not a matrix in the classical sense since it is not a rectangular table of numbers.

Triangular matrices and special parameters connected with them, specifically paradeterminants and parapermanents, are used in many branches of mathematics, see for example [13, 27]. For more information concerning triangular matrices, see [12, 25, 26, 24]. We will cite the most essential results, which are related to this paper.

In [12], the following formulas were given. Let A_n be a triangular matrix and by $\{a_{ij}\}$ let us denote the following expression

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

Then the paradeterminant $\operatorname{ddet}(A_n)$ and parapermanent $\operatorname{pper}(A_n)$ of A_n are

$$\operatorname{ddet}(A_n) = \sum_{r=1}^{n} \sum_{p_1+\cdots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\cdots+p_s, p_1+\cdots+p_{s-1}+1}\}$$

and

$$\operatorname{pper}(A_n) = \sum_{r=1}^{n} \sum_{p_1+\cdots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\cdots+p_s, p_1+\cdots+p_{s-1}+1}\},$$

respectively, where summations are over the set of positive integer solutions of the equality $p_1 + \cdots + p_r = n$.

For $n \geq 1$ we can decompose the paradeterminant and the parapermanent by elements of the last row in the following way (see [12, 24]):

(9) $$\operatorname{ddet}(A_n) = \sum_{s=1}^{n} (-1)^{n-s} \{a_{ns}\} \operatorname{ddet}(A_{s-1}),$$

(10) $$\operatorname{pper}(A_n) = \sum_{s=1}^{n} \{a_{ns}\} \operatorname{pper}(A_{s-1}),$$

where $\operatorname{ddet}(A_0) = 1$, $\operatorname{pper}(A_0) = 1$.

In [28], Zatorsky and Lishchynskyy proved a relation between a paradeterminant of a triangular matrix and a determinant of a special classical matrix, which is almost triangular, known as lower Hessenberg matrix. This is the following relation:

$$(11)\quad \mathrm{ddet}(A_n) = \det \begin{bmatrix} \{a_{11}\} & 1 & 0 & \cdots & 0 & 0 \\ \{a_{21}\} & \{a_{22}\} & 1 & \cdots & 0 & 0 \\ \{a_{31}\} & \{a_{32}\} & \{a_{33}\} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \{a_{n-1,1}\} & \{a_{n-1,2}\} & \{a_{n-1,3}\} & \cdots & \{a_{n-1,n-1}\} & 1 \\ \{a_{n1}\} & \{a_{n2}\} & \{a_{n3}\} & \cdots & \{a_{n,n-1}\} & \{a_{nn}\} \end{bmatrix}.$$

Moreover, there exists a similar connection between parapermanent of a triangular matrix and a permanent of a lower Hessenberg matrix:

$$(12)\quad \mathrm{pper}(A_n) = \mathrm{per} \begin{bmatrix} \{a_{11}\} & 1 & 0 & \cdots & 0 & 0 \\ \{a_{21}\} & \{a_{22}\} & 1 & \cdots & 0 & 0 \\ \{a_{31}\} & \{a_{32}\} & \{a_{33}\} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \{a_{n-1,1}\} & \{a_{n-1,2}\} & \{a_{n-1,3}\} & \cdots & \{a_{n-1,n-1}\} & 1 \\ \{a_{n1}\} & \{a_{n2}\} & \{a_{n3}\} & \cdots & \{a_{n,n-1}\} & \{a_{nn}\} \end{bmatrix}.$$

3. Main results. We are ready to present our results concerning nontrivial connections between special bihyperbolic sequences described in the introduction and paradeterminants of triangular matrices.

Theorem 3.1. *Let $n \geq 0$ be an integer and let*

$$A_{n+1} = \begin{bmatrix} j_1 + 6j_2 + 35j_3 & & & & & \\ -\frac{1}{6} + \frac{1}{6}j_2 + j_3 & 6 & & & & \\ 0 & \frac{1}{6} & 6 & & & \\ 0 & 0 & \frac{1}{6} & 6 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 6 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhB_n = \mathrm{ddet}(A_{n+1})$.

Proof. (By induction with respect to n.)
If $n = 0$, then $\mathrm{ddet}(A_1) = j_1 + 6j_2 + 35j_3 = BhB_0$.

If $n = 1$, then

$$\begin{aligned}
\operatorname{ddet}(A_2) &= \sum_{s=1}^{2}(-1)^{2-s}\{a_{2s}\}\operatorname{ddet}(A_{s-1})\\
&= (-1)^1 \cdot \{a_{21}\}\operatorname{ddet}(A_0) + (-1)^0 \cdot \{a_{22}\}\operatorname{ddet}(A_1)\\
&= (-1)\cdot \prod_{k=1}^{2} a_{2k}\cdot \operatorname{ddet}(A_0) + 1\cdot \prod_{k=2}^{2} a_{2k}\cdot \operatorname{ddet}(A_1)\\
&= (-1)\cdot a_{21}\cdot a_{22}\cdot \operatorname{ddet}(A_0) + 1\cdot a_{22}\cdot \operatorname{ddet}(A_1)\\
&= (-1)\cdot\left(-\frac{1}{6}+\frac{1}{6}j_2+j_3\right)\cdot 6\cdot 1 + 1\cdot 6\cdot (j_1+6j_2+35j_3)\\
&= 1 - j_2 - 6j_3 + 6j_1 + 36j_2 + 210j_3\\
&= 1 + 6j_1 + 35j_2 + 204j_3 = BhB_1.
\end{aligned}$$

Let us assume that for some integer $n \geq 0$ we have $BhB_n = \operatorname{ddet}(A_{n+1})$ and $BhB_{n+1} = \operatorname{ddet}(A_{n+2})$. We will show, that this assumption implies $BhB_{n+2} = \operatorname{ddet}(A_{n+3})$. Using the formula (9), we obtain

$$\begin{aligned}
\operatorname{ddet}(A_{n+3}) &= \sum_{s=1}^{n+3}(-1)^{n+3-s}\{a_{n+3,s}\}\operatorname{ddet}(A_{s-1})\\
&= (-1)^{n+3-1}\cdot\{a_{n+3,1}\}\operatorname{ddet}(A_{1-1})\\
&\quad + \ldots + (-1)^{n+3-(n+1)}\cdot\{a_{n+3,n+1}\}\operatorname{ddet}(A_{n+1-1})\\
&\quad + (-1)^{n+3-(n+2)}\cdot\{a_{n+3,n+2}\}\operatorname{ddet}(A_{n+2-1})\\
&\quad + (-1)^{n+3-(n+3)}\cdot\{a_{n+3,n+3}\}\operatorname{ddet}(A_{n+3-1})\\
&= (-1)^{n+2}\cdot a_{n+3,1}\cdot a_{n+3,2}\cdot\ldots\cdot a_{n+3,n+3}\cdot\operatorname{ddet}(A_0)\\
&\quad + \ldots + 1\cdot a_{n+3,n+1}\cdot a_{n+3,n+2}\cdot\ldots\cdot a_{n+3,n+3}\cdot\operatorname{ddet}(A_n)\\
&\quad + (-1)\cdot a_{n+3,n+2}\cdot a_{n+3,n+3}\cdot\operatorname{ddet}(A_{n+1})\\
&\quad + 1\cdot a_{n+3,n+3}\cdot\operatorname{ddet}(A_{n+2})\\
&= 0 + \ldots + 0 + (-1)\cdot\frac{1}{6}\cdot 6\cdot\operatorname{ddet}(A_{n+1}) + 1\cdot 6\cdot\operatorname{ddet}(A_{n+2})\\
&= -\operatorname{ddet}(A_{n+1}) + 6\operatorname{ddet}(A_{n+2})\\
&= -BhB_n + 6BhB_{n+1} = BhB_{n+2},
\end{aligned}$$

which ends the proof. □

By similar calculations we can prove analogous results for expressing BhC_n, BhM_n, BhH_n as paradeterminants.

Theorem 3.2. *Let $n \geq 0$ be an integer and let*

$$A_{n+1} = \begin{bmatrix} 1+3j_1+17j_2+99j_3 & & & & & \\ \frac{1}{2}+\frac{1}{6}j_1+\frac{1}{2}j_2+\frac{17}{6}j_3 & 6 & & & & \\ 0 & \frac{1}{6} & 6 & & & \\ 0 & 0 & \frac{1}{6} & 6 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 6 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhC_n = \operatorname{ddet}(A_{n+1})$.

Theorem 3.3. *Let $n \geq 0$ be an integer an let*

$$A_{n+1} = \begin{bmatrix} j_1+3j_2+7j_3 & & & & & \\ -\frac{1}{3}+\frac{2}{3}j_2+2j_3 & 3 & & & & \\ 0 & \frac{2}{3} & 3 & & & \\ 0 & 0 & \frac{2}{3} & 3 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 3 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhM_n = \operatorname{ddet}(A_{n+1})$.

Theorem 3.4. *Let $n \geq 0$ be an integer and let*

$$A_{n+1} = \begin{bmatrix} 2+3j_1+5j_2+9j_3 & & & & & \\ 1+\frac{4}{3}j_1+2j_2+\frac{10}{3}j_3 & 3 & & & & \\ 0 & \frac{2}{3} & 3 & & & \\ 0 & 0 & \frac{2}{3} & 3 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 3 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhH_n = \operatorname{ddet}(A_{n+1})$.

Now we show that the same numbers can be also expressed as parapermanents of triangular matrices. We begin the cycle of these theorems with the proof for the numbers BhM_n and the next Theorems 3.6–3.8 can be proved analogously.

Theorem 3.5. *Let $n \geq 0$ be an integer and let*

$$A_{n+1} = \begin{bmatrix} j_1+3j_2+7j_3 & & & & & \\ \frac{1}{3}-\frac{2}{3}j_2-2j_3 & 3 & & & & \\ 0 & -\frac{2}{3} & 3 & & & \\ 0 & 0 & -\frac{2}{3} & 3 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & 3 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhM_n = \operatorname{pper}(A_{n+1})$.

Proof. (By induction with respect to n.)
If $n = 0$, then $\operatorname{pper}(A_1) = j_1 + 3j_2 + 7j_3 = BhM_0$.
If $n = 1$, then

$$\begin{aligned}
\operatorname{pper}(A_2) &= \sum_{s=1}^{2} \{a_{2s}\} \operatorname{pper}(A_{s-1}) = \{a_{21}\} \operatorname{pper}(A_0) + \{a_{22}\} \operatorname{pper}(A_1)\\
&= \prod_{k=1}^{2} a_{2k} \cdot \operatorname{pper}(A_0) + \prod_{k=2}^{2} a_{2k} \cdot \operatorname{pper}(A_1)\\
&= a_{21} \cdot a_{22} \cdot \operatorname{pper}(A_0) + a_{22} \cdot \operatorname{pper}(A_1)\\
&= \left(\frac{1}{3} - \frac{2}{3}j_2 - 2j_3\right) \cdot 3 \cdot 1 + 3 \cdot (j_1 + 3j_2 + 7j_3)\\
&= 1 - 2j_2 - 6j_3 + 3j_1 + 9j_2 + 21j_3\\
&= 1 + 3j_1 + 7j_2 + 15j_3 = BhM_1.
\end{aligned}$$

Let us assume that for some integer $n \geq 0$ we have $BhM_n = \operatorname{pper}(A_{n+1})$ and $BhM_{n+1} = \operatorname{pper}(A_{n+2})$. We will show, that this assumption implies $BhM_{n+2} = \operatorname{pper}(A_{n+3})$. Applying the formula (10), we get

$$\begin{aligned}
\operatorname{pper}(A_{n+3}) &= \sum_{s=1}^{n+3} \{a_{n+3,s}\} \operatorname{pper}(A_{s-1})\\
&= \{a_{n+3,1}\} \operatorname{pper}(A_{1-1}) + \ldots + \{a_{n+3,n+1}\} \operatorname{pper}(A_{n+1-1})\\
&\quad + \{a_{n+3,n+2}\} \operatorname{pper}(A_{n+2-1}) + \{a_{n+3,n+3}\} \operatorname{pper}(A_{n+3-1})\\
&= a_{n+3,1} \cdot a_{n+3,2} \cdot \ldots \cdot a_{n+3,n+3} \cdot \operatorname{pper}(A_0)\\
&\quad + \ldots + a_{n+3,n+1} \cdot a_{n+3,n+2} \cdot \ldots \cdot a_{n+3,n+3} \cdot \operatorname{pper}(A_n)\\
&\quad + a_{n+3,n+2} \cdot a_{n+3,n+3} \cdot \operatorname{pper}(A_{n+1}) + a_{n+3,n+3} \cdot \operatorname{pper}(A_{n+2})\\
&= 0 + \ldots 0 + \left(-\frac{2}{3}\right) \cdot 3 \cdot \operatorname{pper}(A_{n+1}) + 3 \cdot \operatorname{pper}(A_{n+2})\\
&= -2 \operatorname{pper}(A_{n+1}) + 3 \operatorname{pper}(A_{n+2})\\
&= -2BhM_n + 3BhM_{n+1} = BhM_{n+2},
\end{aligned}$$

which end the proof. □

Theorem 3.6. *Let $n \geq 0$ be an integer and let*

$$A_{n+1} = \begin{bmatrix}
j_1 + 6j_2 + 35j_3 & & & & & \\
\frac{1}{6} - \frac{1}{6}j_2 - j_3 & 6 & & & & \\
0 & -\frac{1}{6} & 6 & & & \\
0 & 0 & -\frac{1}{6} & 6 & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & 0 & -\frac{1}{6} & 6
\end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhB_n = \operatorname{pper}(A_{n+1})$.

Theorem 3.7. *Let $n \geq 0$ be an integer and let*

$$A_{n+1} = \begin{bmatrix} 1+3j_1+17j_2+99j_3 & & & & & \\ -\frac{1}{2}-\frac{1}{6}j_1-\frac{1}{2}j_2-\frac{17}{6}j_3 & 6 & & & & \\ 0 & -\frac{1}{6} & 6 & & & \\ 0 & 0 & -\frac{1}{6} & 6 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -\frac{1}{6} & 6 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhC_n = \mathrm{pper}(A_{n+1})$.

Theorem 3.8. *Let $n \geq 0$ be an integer and let*

$$A_{n+1} = \begin{bmatrix} 2+3j_1+5j_2+9j_3 & & & & & \\ -1-\frac{4}{3}j_1-2j_2-\frac{10}{3}j_3 & 3 & & & & \\ 0 & -\frac{2}{3} & 3 & & & \\ 0 & 0 & -\frac{2}{3} & 3 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & 3 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhH_n = \mathrm{pper}(A_{n+1})$.

By formulas (11) and (12), using Theorems 3.1–3.8, we can directly obtain corollaries, which express terms of sequences (1)–(4) as determinants and permanents of classical matrices, being lower Hessenberg matrices.

Corollary 3.9. *Let $n \geq 0$ be an integer. Then*

$$BhB_n = \det \begin{bmatrix} j_1+6j_2+35j_3 & 1 & 0 & \cdots & 0 & 0 \\ -1+j_2+6j_3 & 6 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 6 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 6 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$BhC_n = \det \begin{bmatrix} 1+3j_1+17j_2+99j_3 & 1 & 0 & \cdots & 0 & 0 \\ 3+j_1+3j_2+17j_3 & 6 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 6 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 6 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$BhM_n = \det\begin{bmatrix} j_1+3j_2+7j_3 & 1 & 0 & 0 & 0 & 0\\ -1+2j_2+6j_3 & 3 & 1 & 0 & 0 & 0\\ 0 & 2 & 3 & 1 & 0 & 0\\ 0 & 0 & 2 & 3 & 1 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 2 & 3 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$BhH_n = \det\begin{bmatrix} 2+3j_1+5j_2+9j_3 & 1 & 0 & 0 & 0 & 0\\ 3+4j_1+6j_2+10j_3 & 3 & 1 & 0 & 0 & 0\\ 0 & 2 & 3 & 1 & 0 & 0\\ 0 & 0 & 2 & 3 & 1 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 2 & 3 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Corollary 3.10. *Let $n \geq 0$ be an integer. Then*

$$BhB_n = \operatorname{per}\begin{bmatrix} j_1+6j_2+35j_3 & 1 & 0 & \cdots & 0 & 0\\ 1-j_2-6j_3 & 6 & 1 & \cdots & 0 & 0\\ 0 & -1 & 6 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -1 & 6 & 1\\ 0 & 0 & 0 & \cdots & -1 & 6 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$BhC_n = \operatorname{per}\begin{bmatrix} 1+3j_1+17j_2+99j_3 & 1 & 0 & \cdots & 0 & 0\\ -3-j_1-3j_2-17j_3 & 6 & 1 & \cdots & 0 & 0\\ 0 & -1 & 6 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -1 & 6 & 1\\ 0 & 0 & 0 & \cdots & -1 & 6 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$BhM_n = \operatorname{per}\begin{bmatrix} j_1+3j_2+7j_3 & 1 & 0 & 0 & 0 & 0\\ 1-2j_2-6j_3 & 3 & 1 & 0 & 0 & 0\\ 0 & -2 & 3 & 1 & 0 & 0\\ 0 & 0 & -2 & 3 & 1 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & -2 & 3 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$BhH_n = \operatorname{per}\begin{bmatrix} 2+3j_1+5j_2+9j_3 & 1 & 0 & 0 & 0 & 0\\ -3-4j_1-6j_2-10j_3 & 3 & 1 & 0 & 0 & 0\\ 0 & -2 & 3 & 1 & 0 & 0\\ 0 & 0 & -2 & 3 & 1 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & -2 & 3 \end{bmatrix}_{(n+1)\times(n+1)}.$$

4. Examples. In this subsection we show how Corollary 3.10 and Theorem 3.1 work in specific cases, namely we express the number BhM_3 as a permanent of a matrix and the number BhB_3 as a paradeterminant of a triangular matrix. In both cases, the permanent and the paradeterminant are decomposed by elements of the last row.

$$\begin{aligned}
&\operatorname{per}\begin{bmatrix} j_1+3j_2+7j_3 & 1 & 0 & 0\\ 1-2j_2-6j_3 & 3 & 1 & 0\\ 0 & -2 & 3 & 1\\ 0 & 0 & -2 & 3\end{bmatrix}_{4\times 4}\\
&= 0\cdot\operatorname{per}\begin{bmatrix} 1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 3 & 1\end{bmatrix}_{3\times 3} + 0\cdot\operatorname{per}\begin{bmatrix} j_1+3j_2+7j_3 & 0 & 0\\ 1-2j_2-6j_3 & 1 & 0\\ 0 & 3 & 1\end{bmatrix}_{3\times 3}\\
&= (-2)\cdot\operatorname{per}\begin{bmatrix} j_1+3j_2+7j_3 & 1 & 0\\ 1-2j_2-6j_3 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}_{3\times 3} + 3\cdot\operatorname{per}\begin{bmatrix} j_1+3j_2+7j_3 & 1 & 0\\ 1-2j_2-6j_3 & 3 & 1\\ 0 & -2 & 3\end{bmatrix}_{3\times 3}\\
&= (-2)\cdot((j_1+3j_2+7j_3)\cdot 3+(1-2j_2-6j_3))\\
&\quad + 3\cdot((j_1+3j_2+7j_3)\cdot 9-2\cdot(j_1+3j_2+7j_3)+3\cdot(1-2j_2-6j_3))\\
&= (-2)\cdot(1+3j_1+7j_2+15j_3)+3\cdot(3+7j_1+15j_2+31j_3)\\
&= 7+15j_1+31j_2+63j_3 = BhM_3.
\end{aligned}$$

$$\begin{aligned}
&\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3 & & & \\ -\frac{1}{6}+\frac{1}{6}j_2+j_3 & 6 & & \\ 0 & \frac{1}{6} & 6 & \\ 0 & 0 & \frac{1}{6} & 6\end{bmatrix}_{4\times 4}\\
&= (-1)^3\cdot\left(0\cdot 0\cdot\frac{1}{6}\cdot 6\right)\cdot 1 + (-1)^2\cdot\left(0\cdot\frac{1}{6}\cdot 6\right)\cdot\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3\end{bmatrix}_{1\times 1}\\
&\quad + (-1)^1\cdot\left(\frac{1}{6}\cdot 6\right)\cdot\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3 & \\ -\frac{1}{6}+\frac{1}{6}j_2+j_3 & 6\end{bmatrix}_{2\times 2}\\
&\quad + (-1)^0\cdot 6\cdot\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3 & & \\ -\frac{1}{6}+\frac{1}{6}j_2+j_3 & 6 & \\ 0 & \frac{1}{6} & 6\end{bmatrix}_{3\times 3}\\
&= 0+0-\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3 & \\ -\frac{1}{6}+\frac{1}{6}j_2+j_3 & 6\end{bmatrix}_{2\times 2} + 6\cdot\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3 & & \\ -\frac{1}{6}+\frac{1}{6}j_2+j_3 & 6 & \\ 0 & \frac{1}{6} & 6\end{bmatrix}_{3\times 3}\\
&= -\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3 & \\ -\frac{1}{6}+\frac{1}{6}j_2+j_3 & 6\end{bmatrix}_{2\times 2} + 6\cdot(-1)^1\cdot\left(\frac{1}{6}\cdot 6\right)\cdot\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3\end{bmatrix}_{1\times 1}\\
&\quad + 6\cdot(-1)^0\cdot 6\cdot\operatorname{ddet}\begin{bmatrix} j_1+6j_2+35j_3 & \\ -\frac{1}{6}+\frac{1}{6}j_2+j_3 & 6\end{bmatrix}_{2\times 2}
\end{aligned}$$

$$= 35 \cdot \operatorname{ddet} \begin{bmatrix} j_1 + 6j_2 + 35j_3 & \\ -\frac{1}{6} + \frac{1}{6}j_2 + j_3 & 6 \end{bmatrix}_{2\times 2} - 6 \cdot \operatorname{ddet} \left[j_1 + 6j_2 + 35j_3 \right]_{1\times 1}$$

$$= 35 \cdot \left((-1)^1 \cdot \left(-\frac{1}{6} + \frac{1}{6}j_2 + j_3 \right) \cdot 6 \cdot 1 + (-1)^0 \cdot 6 \cdot \operatorname{ddet} \left[j_1 + 6j_2 + 35j_3 \right]_{1\times 1} \right)$$

$$+ \left(-6 \cdot \operatorname{ddet} \left[j_1 + 6j_2 + 35j_3 \right]_{1\times 1} \right)$$

$$= 35 \cdot (1 - j_2 - 6j_3) + 204 \cdot \operatorname{ddet} \left[j_1 + 6j_2 + 35j_3 \right]_{1\times 1}$$

$$= 35 - 35j_2 - 210j_3 + 204 \cdot (j_1 + 6j_2 + 35j_3)$$

$$= 35 + 204j_1 + 1189j_2 + 6930j_3$$

$$= BhB_3.$$

5. Conclusions. In our paper we have considered special bihyperbolic numbers given by linear recurrence of the second order. We have shown that these numbers can be expressed as certain parameters of specific triangular tables or matrices. The obtained results may be a starting point for the search of further relations between matrix theory and bihyperbolic numbers defined by linear or nonlinear recurrence equations of order $k \geq 3$, for example the well-known Padovan recurrence given by the formula $P(n) = P(n-2) + P(n-3)$, where $P(0) = P(1) = P(2) = 1$.

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Paweł Bednarz
Rzeszow University of Technology
The Faculty of Mathematics
and Applied Physics
Department of Discrete Mathematics
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: `pbednarz@prz.edu.pl`

Anna Kosiorowska
Rzeszow University of Technology
The Faculty of Mathematics
and Applied Physics
Department of Discrete Mathematics
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: `a.kosiorowsk@prz.edu.pl`

Adrian Michalski
Rzeszow University of Technology
The Faculty of Mathematics and Applied Physics
Department of Discrete Mathematics
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: a.michalski@prz.edu.pl

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