doi: 10.17951/a.2024.78.1.75-86

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXVIII, NO. 1, 2024 SECTIO A 75–86

JAN KUREK and WŁODZIMIERZ M. MIKULSKI

The Euler-like operators on tuples of Lagrangians and functions on bases

Abstract. Let $\mathcal{FM}_{m,n}$ denote the category of fibered manifolds with m-dimensional bases and n-dimensional fibres and their fibered diffeomorphisms onto open images. We describe all $\mathcal{FM}_{m,n}$-natural operators C transforming tuples (λ, g) of Lagrangians $\lambda : J^sY \to \bigwedge^m T^*M$ (or formal Lagrangians $\lambda : J^sY \to V^*J^sY \otimes \bigwedge^m T^*M$) on $\mathcal{FM}_{m,n}$-objects $Y \to M$ and functions $g : M \to \mathbf{R}$ into Euler maps $C(\lambda, g) : J^{2s}Y \to V^*Y \otimes \bigwedge^m T^*M$ on Y. The most important example of such C is the Euler operator E (from the variational calculus) (or the formal Euler operator $\mathbf{E}$) treated as the operator in question depending only on Lagrangians (or formal Lagrangians).

1. Introduction. All manifolds considered in this paper are assumed to be finite dimensional, without boundary and smooth (i.e. of class $\mathcal{C}^\infty$). Mappings between manifolds are assumed to be smooth (of class $\mathcal{C}^\infty$).

Given a fibred manifold $Y \to M$, we have the s-jet prolongation J^sY of $Y \to M$ and the obvious jet projection $\pi^{2s}_s : J^{2s}Y \to J^sY$ for any positive integer s. We also have the vertical bundle $VY \to Y$, its dual bundle $V^*Y \to Y$, the cotangent bundle T^*M and its mth inner product $\bigwedge^m T^*M$. Given fibred manifolds $Z_1 \to M$ and $Z_2 \to M$ with the same basis M, let $\mathcal{C}^\infty_M(Z_1, Z_2)$ denote the space of all base preserving fibred maps of Z_1 into Z_2. Let m be the dimension of the base M of Y. Elements from the space $\mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ are called (sth order) Lagrangians on $Y \to M$. Elements

2010 *Mathematics Subject Classification.* 58A20, 58E30.

Key words and phrases. Fibered manifolds, Lagrangian, Euler map, natural operator, Euler operator, formal Euler operator.

from the space $\mathcal{C}^\infty_{J^sY}(J^sY, V^*J^sY \otimes \bigwedge^m T^*M)$ are called (sth order) formal Lagrangians on $Y \to M$. Elements from the space $\mathcal{C}^\infty_Y(J^qY, V^*Y \otimes \bigwedge^m T^*M)$ are called Euler maps on $Y \to M$. The concept of natural operators can be found in [3].

By Proposition 49.3 of [3], any sth order Lagrangian $\lambda : J^sY \to \bigwedge^m T^*M$ on a fibered manifold $Y \to M$ induces canonically the Euler map $E(\lambda) : J^{2s}Y \to V^*Y \otimes \bigwedge^m T^*M$. So, we have the so-called Euler operator

$$E : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M\right).$$

In [1] (see [4]), I. Kolář proved the following:

Theorem 1.1. *Let m, n, s be positive integers. If $m \geq 2$, then any regular, π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M\right)$$

is of the form cE, $c \in \mathbf{R}$, where E is the Euler operator.

Here and later $\mathcal{FM}_{m,n}$ denotes the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred diffeomorphisms onto open images.

In our paper, we study the more general problem how a tuple (λ, g) of a Lagrangian $\lambda \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ and a map $g \in \mathcal{C}^\infty(M, \mathbf{R})$ can induce canonically an Euler map $C(\lambda, g) \in \mathcal{C}^\infty_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$.

Namely, in our paper, if $m \geq 2$, we describe all regular and π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operators

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M, \mathbf{R}) \to \mathcal{C}^\infty_Y\left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M\right).$$

Further, in [2] (see [4]), I. Kolář introduced the so-called formal Euler operator

$$\mathbf{E} : \mathcal{C}^\infty_{J^sY}\left(J^sY, V^*J^sY \otimes \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M\right)$$

for all $\mathcal{FM}_{m,n}$-objects $Y \to M$. In [6], we proved:

Theorem 1.2. *Let m, n, s be positive integers. Then any regular, π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_{J^sY}\left(J^sY, V^*J^sY \otimes \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M\right)$$

is of the form $c\mathbf{E}$, $c \in \mathbf{R}$, where $\mathbf{E}$ is the formal Euler operator.

In our paper, if $m \geq 2$, we also describe all regular, π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operators

$$C:\mathcal{C}^\infty_{J^sY}\left(J^sY, V^*J^sY\otimes\bigwedge^m T^*M\right)\times\mathcal{C}^\infty(M,\mathbf{R})\to\mathcal{C}^\infty_Y\left(J^{2s}Y, V^*Y\otimes\bigwedge^m T^*M\right).$$

2. The main results.

Example 2.1. Let $l=0,1,\dots,s$. We define $E^{(l)}(\lambda,g):J^{2s}Y\to V^*Y\otimes\bigwedge^m T^*M$ by

$$E^{(l)}(\lambda,g)_{|j^{2s}_{x_o}\sigma}:=E((-1)^l(g-g(x_o))^l\cdot\lambda)_{|j^{2s}_{x_o}\sigma}$$

for any $\lambda\in\mathcal{C}^\infty_M(J^sY,\bigwedge^m T^*M)$ on an $\mathcal{FM}_{m,n}$-object $Y\to M$, any $g\in\mathcal{C}^\infty(M,\mathbf{R})$, any $j^{2s}_{x_o}\sigma\in J^{2s}_{x_o}Y$ and any $x_o\in M$, where E is the Euler operator. Thus we have the corresponding $\mathcal{FM}_{m,n}$-natural operator

$$E^{(l)}:\mathcal{C}^\infty_M\left(J^sY,\bigwedge^m T^*M\right)\times\mathcal{C}^\infty(M,\mathbf{R})\to\mathcal{C}^\infty_Y\left(J^{2s}Y,V^*Y\otimes\bigwedge^m T^*M\right).$$

We call $E^{(l)}$ the lth modification of E. Clearly, $E^{(0)}=E$.

The first main result of our paper is the following:

Theorem 2.2. *Let m,n,s be positive integers. If $m\geq 2$, then any regular, π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural (i.e. invariant with respect to $\mathcal{FM}_{m,n}$-maps) operator*

$$C:\mathcal{C}^\infty_M\left(J^sY,\bigwedge^m T^*M\right)\times\mathcal{C}^\infty(M,\mathbf{R})\to\mathcal{C}^\infty_Y\left(J^{2s}Y,V^*Y\otimes\bigwedge^m T^*M\right)$$

is $C=\sum_{l=0}^s h_l\cdot E^{(l)}$ for some (uniquely determined by C) maps $h_l:\mathbf{R}\to\mathbf{R}$, $l=0,\dots,s$, where the multiplication $h\cdot C$ is defined by

$$(h\cdot C)(\lambda,g)_{|j^{2s}_{x_o}\sigma}=h(g(x_o))\cdot C(\lambda,g)_{|j^{2s}_{x_o}\sigma}$$

for any $h:\mathbf{R}\to\mathbf{R}$, any C in question and any λ, g, $j^{2s}_{x_o}\sigma$ as above.

In other words, the space of all C in question is the free $(s+1)$-dimensional $\mathcal{C}^\infty(\mathbf{R})$-module and the operators $E^{(l)}$ for $l=0,1,\dots,s$ form the basis in this module.

Remark 2.3. The $\mathcal{FM}_{m,n}$-invariance of C means that for any $\mathcal{FM}_{m,n}$-map $f:Y\to Y_1$, Lagrangians $\lambda\in\mathcal{C}^\infty_M(J^sY,\bigwedge^m T^*M)$, $\lambda_1\in\mathcal{C}^\infty_{M_1}(J^sY_1,\bigwedge^m T^*M_1)$ and maps $g:M\to\mathbf{R}$ and $g_1:M_1\to\mathbf{R}$, if λ and λ_1 are f-related and g and g_1 are f-related, then so are $C(\lambda)$ and $C(\lambda_1)$. The π^{2s}_s-locality of C means that $C(\lambda,g)_\rho$ depends on $\mathrm{germ}_{\pi^{2s}_s(\rho)}(\lambda,g)$ for any $\rho\in J^{2s}Y$ and $\lambda\in\mathcal{C}^\infty_M(J^sY,\bigwedge^m T^*M)$ and any $g\in\mathcal{C}^\infty(M,\mathbf{R})$. The regularity of C means that C transforms smoothly parametrized families of tuples of Lagrangians and maps on the bases into smoothly parametrized families of Euler maps.

Clearly, Theorem 1.1 is a simple consequence of Theorem 2.2. The proof of Theorem 2.2 will be given in Section 4.

Remark 2.4. If $m=1$, Theorem 2.2 does not hold. Indeed, in [1], I. Kolář constructed a regular, π^2_1-local and $\mathcal{FM}_{1,n}$-natural operator $W:\mathcal{C}^\infty_M(J^1Y,T^*M)\to\mathcal{C}^\infty_Y(J^2Y,V^*Y\otimes\bigwedge^m T^*M)$ which is not cE. Suppose $W=\sum_{l=0}^s h_i\cdot E^{(l)}$. Then $W(\lambda)=W(\lambda,1)=h_0(1)E(\lambda)$, i.e. $W=h_0(1)E$ which is a contradiction.

Example 2.5. Let $l=0,\dots,s$. By the same way as in the previous example, we define $\mathbf{E}^{(l)}(\lambda,g):J^{2s}Y\to V^*Y\otimes\bigwedge^m T^*M$ by

$$\mathbf{E}^{(l)}(\lambda,g)_{|j^{2s}_{x_o}\sigma}:=\mathbf{E}((-1)^l(g-g(x_o))^l\cdot\lambda)_{|j^{2s}_{x_o}\sigma}$$

for any $\lambda\in\mathcal{C}^\infty_M(J^sY,V^*J^sY\otimes\bigwedge^m T^*M)$, any $g\in\mathcal{C}^\infty(M,\mathbf{R})$, any $j^{2s}_{x_o}\sigma\in J^{2s}_{x_o}Y$ and any $x_o\in M$, where $\mathbf{E}$ is the formal Euler operator. Thus we have the corresponding $\mathcal{FM}_{m,n}$-natural operator

$$\mathbf{E}^{(l)}:\mathcal{C}^\infty_{J^sY}\left(J^sY,V^*J^sY\otimes\bigwedge^m T^*M\right)\times\mathcal{C}^\infty(M,\mathbf{R})\to\mathcal{C}^\infty_Y\left(J^{2s}Y,V^*Y\otimes\bigwedge^m T^*M\right).$$

We call $\mathbf{E}^{(l)}$ the lth modification of $\mathbf{E}$. Clearly, $\mathbf{E}^{(0)}=\mathbf{E}$.

The second main result of our paper is the following:

Theorem 2.6. *Let m,n,s be positive integers. If $m\geq 2$, then any regular, π^{2s}_s-local and $\mathcal{FM}_{m,n}$-natural operator*

$$C:\mathcal{C}^\infty_{J^sY}\left(J^sY,V^*J^sY\otimes\bigwedge^m T^*M\right)\times\mathcal{C}^\infty(M,\mathbf{R})\to\mathcal{C}^\infty_Y\left(J^{2s}Y,V^*Y\otimes\bigwedge^m T^*M\right)$$

is $C=\sum_{l=0}^s h_l\cdot\mathbf{E}^{(l)}$ for some (uniquely determined by C) maps $h_l:\mathbf{R}\to\mathbf{R}$, $l=0,\dots,s$, where $h\cdot C$ is quite similar as in the previous theorem.

So, the space of all C in question is also the free $(s+1)$-dimensional $\mathcal{C}^\infty(\mathbf{R})$-module and the operators $\mathbf{E}^{(l)}$ for $l=0,1,\dots,s$ form the basis in this module.

Clearly, Theorem 1.2 for $m\geq 2$ is a simple consequence of Theorem 2.6. The schema of the proof of Theorem 2.6 will be presented in Section 5.

3. Preparation. From now on, let $\mathbf{N}=\{0,1,2,\dots\}$ be the set of non-negative integers and let $\mathbf{R}^{m,n}$ denotes the trivial (affine) bundle $\mathbf{R}^m\times\mathbf{R}^n\to\mathbf{R}^m$ and let $x^1,\dots,x^m,y^1,\dots,y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$. Let $dx^\mu=dx^1\wedge\dots\wedge dx^m$. Given $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbf{N}^m$, let $x^\alpha:=(x^1)^{\alpha_1}\cdot\ldots\cdot(x^m)^{\alpha_m}$. Given $i=1,\dots,m$, let $1_i:=(0,\dots,0,1,0,\dots,0)\in\mathbf{N}^m$ where 1 is in the ith position.

Given a fibred manifold $Y\to M$, the s-jet prolongation J^sY of $Y\to M$ is the fibred manifold (with the base M) of all s-jets $j^s_x\sigma$ at $x\in M$ of local sections $\sigma:M\to Y$ of $Y\to M$. If $Y\to M$ and $Y^1\to M_1$ are fibred manifolds with m-dimensional bases M and M_1 and $f:Y\to Y^1$ is a fibred map with the base map $\underline{f}:M\to M_1$ being local diffeomorphism, then we have the fibred map $J^sf: J^sY \to J^sY_1$ defined by $J^rf(j^s_{x_o}\sigma)=j^s_{\underline{f}(x_o)}(f\circ\sigma\circ\underline{f}^{-1})$, $j^s_{x_o}\sigma\in J^s_{x_o}Y$, $x_o\in M$.

Let $((x^i),(y^j_\alpha))$ be the induced coordinates on $J^s(\mathbf{R}^{m,n})$, where $i=1,\dots,m$, $j=1,\dots,n$ and $\alpha=(\alpha_1,\dots\alpha_m)\in\mathbf{N}^m$ are such that $|\alpha|=\alpha_1+\dots+\alpha_m\le s$. We remind that

$$x^i(j^s_{x_o}\sigma)=x^i_o \quad\text{and}\quad y^j_\alpha(j^s_{x_o}\sigma)=(\partial_\alpha\sigma^j)(x_o)$$

for any $j^s_{x_o}\sigma=j^s_{x_o}(\sigma^1,\dots,\sigma^n)\in J^s_{x_o}(\mathbf{R}^{m,n})=J^s_{x_o}(\mathbf{R}^m,\mathbf{R}^n)$, $x_o\in\mathbf{R}^m$, where ∂_α is the iterated partial derivative as indicated multiplied by $\frac{1}{\alpha!}$.

Lemma 3.1. *Let $i=1,\dots,m$, $j=1,\dots,n$ and $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbf{N}^m$ be such that $|\alpha|\le s$.*

(i) For any $\tau=(\tau^1,\dots,\tau^n)\in(\mathbf{R}\setminus\{0\})^n$, we have

$$(J^s\psi_\tau)_*y^j_\alpha=\tau^j y^j_\alpha\,,$$

where $\psi_\tau=(x^1,\dots,x^m,\frac{1}{\tau^1}y^1,\dots,\frac{1}{\tau^n}y^n)$ is the $\mathcal{FM}_{m,n}$-map.

(ii) For any $t\in\mathbf{R}\setminus\{0\}$, we have

$$(J^s\varphi^i_t)_*y^j_\alpha=t^{-\alpha_i}y^j_\alpha\,,$$

where $\varphi^i_t=(x^1,\dots,\frac{1}{t}x^i,\dots,x^m,y^1,\dots,y^n)$ is the $\mathcal{FM}_{m,n}$-map.

(iii) If $\alpha_i\neq0$, we have

$$(J^s\psi^{(i)})_*y^1_\alpha=y^1_\alpha+x^iy^1_\alpha+y^1_{\alpha-1_i}\,,$$

where $\psi^{(i)}=(x^1,\dots,x^m,y^1+x^iy^1,y^2,\dots,y^n)^{-1}$ is the $\mathcal{FM}_{m,n}$-map (defined over $0\in\mathbf{R}^m$).

(iv) If $\alpha_1\neq0$, we have

$$(J^s\chi_t)_*y^1_{\alpha-1_1+1_2}=y^1_{\alpha-1_1+1_2}+c_1ty^1_\alpha+\dots+c_{\alpha_2+1}t^{\alpha_2+1}y^1_{(\alpha_1+\alpha_2,0,\alpha_3,\dots,\alpha_m)}$$

for some $c_1,\dots\in\mathbf{R}$ with $c_1\neq0$, where $\chi_t=(x^1+tx^2,x^2,\dots,x^m,y^1,\dots,y^n)$ is the $\mathcal{FM}_{m,n}$-map (defined if $m\ge2$).

Proof. We prove (iii) and (iv), only. The proofs of the other parts are similar.

Ad (iii). If $\alpha_i\neq0$, we have

$$\begin{aligned}y^1_\alpha\circ J^s(\psi^{(i)})^{-1}(j^s_{x_o}\sigma)&=\partial_\alpha(\sigma^1+x^i\sigma^1)(x_o)\\&=\partial_\alpha\sigma^1(x_o)+x^i_o\partial_\alpha\sigma^1(x_o)+\partial_{\alpha-1_i}\sigma^1(x_o)\end{aligned}$$

for any $j^s_{x_o}\sigma\in J^s(\mathbf{R}^{m,n})=J^s(\mathbf{R}^m,\mathbf{R}^n)$. Then

$$((J^s\psi^{(i)})_*y^1_\alpha)(j^s_{x_o}\sigma)=y^1_\alpha\circ J^s(\psi^{(i)})^{-1}(j^s_{x_o}\sigma)=(y^1_\alpha+x^iy^1_\alpha+y^1_{\alpha-1_i})(j^s_{x_o}\sigma)$$

for any $j^s_{x_o}\sigma\in J^s(\mathbf{R}^{m,n})=J^s(\mathbf{R}^m,\mathbf{R}^n)$.

Ad (iv). Similarly, we have

$$y^1_{\alpha-1_1+1_2}\circ(J^s\chi_t)^{-1}(j^s_{x_o}\sigma)=\partial_{\alpha-1_1+1_2}(\sigma^1(x^1+tx_2,x^2,\dots,x^m))((\underline{\chi_t})^{-1}(x_o))$$
$$=\partial_{\alpha-1_1+1_2}\sigma^1(x_o)+c_1t\partial_\alpha\sigma^1(x_o)+c_2t^2\partial_{\alpha+1_1-1_2}\sigma^1(x_o)+(\dots)\,,$$

where $\underline{\chi_t}=(x^1+tx^2,x^2,\dots,x^m)$ is the base map of χ_t. Then

$$((J^s\chi_t)_*y^1_{\alpha-1_1+1_2})(j^s_{x_o}\sigma)=y^1_{\alpha-1_1+1_2}\circ(J^s\chi_t)^{-1}(j^s_{x_o}\sigma)$$
$$=(y^1_{\alpha-1_1+1_2}+c_1ty^1_\alpha+\dots)(j^s_{x_o}\sigma)\,. \qquad \square$$

4. Proof of Theorem 2.2. Because of the invariance of C with respect to the $\mathcal{FM}_{m,n}$-charts, C is determined by the collection of values

$$\langle C(\lambda,g)_\rho,v\rangle\in\bigwedge^m T^*_0\mathbf{R}^m$$

for all $\lambda\in\mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}),\bigwedge^m T^*\mathbf{R}^m)$, $v\in T_0\mathbf{R}^n=V_{(0,0)}\mathbf{R}^{m,n}$, $\rho=j^{2s}_0(\sigma)\in J^{2s}_0(\mathbf{R}^m,\mathbf{R}^n)=J^{2s}_0(\mathbf{R}^{m,n})$ and $g:\mathbf{R}^m\to\mathbf{R}$. (Here and from now on the phrase "C is determined by..." means that if C' is another operator in question giving the same collection of values as C, then $C=C'$.)

For any element $\rho=j^{2s}_0(\sigma)\in J^{2s}_0(\mathbf{R}^m,\mathbf{R}^n)=J^{2s}_0(\mathbf{R}^{m,n})$, there exists an $\mathcal{FM}_{m,n}$-map $\nu:\mathbf{R}^{m,n}\to\mathbf{R}^{m,n}$ transforming $j^{2s}_0(\sigma)$ into $\theta:=j^{2s}_0(0)\in J^{2s}_0(\mathbf{R}^m,\mathbf{R}^n)=J^{2s}_0(\mathbf{R}^{m,n})$. (Indeed, we can choose $\nu:=(x,y-\sigma(x))$, where $x=(x^1,\dots,x^m)$ and $y=(y^1,\dots,y^n)$.) So, we can additionally assume that $\rho=\theta$.

Because of the regularity of C, we can assume that $d_0g\neq0$. Then using the invariance of C with respect to a $(0,0)$-preserving $\mathcal{FM}_{m,n}$-map of the form $(\varphi(x),y^1,\dots,y^n)$ (it preserves θ), we may additionally assume that $g=x^m+c$, where c is an arbitrary real number.

We can obviously assume that $v\neq0$. Then using the invariance of C with respect to a respective $\mathcal{FM}_{m,n}$-map of the form $\mathrm{id}_{\mathbf{R}^m}\times\phi$ for a linear isomorphism $\phi:\mathbf{R}^n\to\mathbf{R}^n$, we can additionally assume that $v=\frac{\partial}{\partial y^1}{}_{|(0,0)}$ (because ϕ preserves θ and g).

Further, we can write $\lambda=L((x^i),(y^j_\alpha))dx^\mu+f(x^1,\dots,x^m)dx^\mu$, where L and f are arbitrary real valued maps with $L((x^i),(0))=0$. By the regularity of C, we can assume that $f(0)\neq0$. Then, using the invariance of C with respect to $\mathcal{FM}_{m,n}$-map $b=(F(x^1,..,x^m),x^2,\dots,x^m,y^1,\dots,y^n)^{-1}$, where $\frac{\partial}{\partial x^1}F=f$ and $F(0,x^2,\dots,x^m)=0$, we may additionally assume that $f=1$ because b preserves θ, g (as $m\geq2$) and $\frac{\partial}{\partial y^1}{}_{|(0,0)}$ and it sends dx^μ into fdx^μ. Consequently, we can write $\lambda=L((x^i),(y^j_\alpha))dx^\mu+dx^\mu$, where L is an arbitrary real valued map with $L((x^i),(0))=0$.

Further, because of the π^{2s}_s-locality of C, using the main result of [5], we may additionally assume that L is a arbitrary polynomial in $((x^i),(y^j_\alpha))$ of degree $\leq q$, where q is an arbitrary positive integer.

Further, by the invariance of C with respect to $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1} y^1, \dots, \frac{1}{\tau^n} y^n)$ being $\mathcal{FM}_{m,n}$-map for any $(\tau^1, \dots, \tau^n) \in (\mathbf{R} \setminus \{0\})^n$, we get the homogeneity condition

$$\left\langle C(L((x^i), (\tau^j y^j_\alpha)) dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \\ = \tau^1 \left\langle C(L((x^i), (y^j_\alpha)) dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle,$$

see Lemma 3.1 (i). Then by the homogeneous function theorem ([3]), we conclude that $\left\langle C(L dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$ depends linearly on L and C is determined by the collection of values

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

for $\alpha, \beta \in \mathbf{N}^m$ with $|\beta| \le q$ and $|\alpha| \le s$.

Further, by the invariance of C with respect to $\varphi^i_t = (x^1, \dots, \frac{1}{t} x^i, \dots, x^m, y^1, \dots, y^n)$ being $\mathcal{FM}_{m,n}$-map for any $t \in \mathbf{R} \setminus \{0\}$ and any $i = 1, \dots, m$ and using the fact that $\left\langle C(L dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$ depends linearly on L, we get the condition

$$t^{\beta_i - \alpha_i} \left\langle C(x^\beta y^1_\alpha dx^\mu + t dx^\mu, t^{\delta_{im}} x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \\ = \left\langle C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

because φ^i_t preserves C, θ and $\frac{\partial}{\partial y^1}_{|(0,0)}$ and it sends x^β into $t^{\beta_i} x^\beta$, x^m into $t^{\delta_{im}} x^m$ (the Kronecker delta), y^1_α into $t^{-\alpha_i} y^1_\alpha$ and dx^μ into $t dx^\mu$, see Lemma 3.1 (ii). Then putting $t \to 0$, we get the condition

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle = 0$$

for any $\alpha, \beta \in \mathbf{N}^m$ with both $|\alpha| \le s$ and $\beta_i > \alpha_i$ for some $i = 1, \dots, m$. Consequently, C is determined by the collection of values

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \in \bigwedge^m T^*_0 \mathbf{R}^m$$

for all $c \in \mathbf{R}$ and all $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \le s$ and $\beta_1 \le \alpha_1, \dots, \beta_m \le \alpha_m$.

Consider $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \le s$ and $\beta_1 \le \alpha_1, \dots, \beta_m \le \alpha_m$. Assume that $\beta \ne (0)$. For example, let $\beta_i \ne 0$ for some $i = 1, \dots, m$. Using the invariance of

C with respect to $\psi^{(i)} = (x^1, \dots, x^m, y^1 + x^i y^1, y^2, \dots, y^n)^{-1}$ (being $\mathcal{FM}_{m,n}$-map defined over some neighborhood of $0 \in \mathbf{R}^m$), we get

$$\left\langle C(x^{\beta-1_i}(y^1_\alpha + x^i y^1_\alpha + y^1_{\alpha-1_i})dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle = \left\langle C(x^{\beta-1_i} y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

because $\psi^{(i)}$ preserves C, $x^{\beta-1_i}$, θ, $\frac{\partial}{\partial y^1}_{|(0,0)}$, dx^μ and $x^m + c$ and it sends y^1_α into $y^1_\alpha + x^i y^1_\alpha + y^1_{\alpha-1_i}$, see Lemma 3.1 (iii). Then

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle = -\left\langle C(x^{\beta-1_i} y^1_{\alpha-1_i} dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

because $\left\langle C(L dx^\mu + dx^\mu)_\theta, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$ depends linearly on L. Repeating this process, we get

$$\left\langle C(x^\beta y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle = (-1)^{|\beta|} \left\langle C(y^1_{(\alpha-\beta)} dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle .$$

Consequently, C is determined by the collection of values

$$\left\langle C(y^1_\alpha dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \in \bigwedge^m T^*_0 \mathbf{R}^m$$

for all $c \in \mathbf{R}$ and all $\alpha \in \mathbf{N}^m$ with $|\alpha| \le s$.

Let $\alpha \in \mathbf{N}^m$, where $|\alpha| \le s$, and assume that $\alpha_i \ne 0$ for some $i = 1, \dots, m-1$. For example, let $\alpha_1 \ne 0$. For any $t \in \mathbf{R}$, the $\mathcal{FM}_{m,n}$-map $\chi_t = (x^1 + tx^2, x^2, \dots, x^m, y^1, \dots, y^n)$ (defined if $m \ge 2$) preserves dx^μ, θ, $\frac{\partial}{\partial y^1}_{|(0,0)}$ and $x^m + c$ and it sends $y^1_{\alpha-1_1+1_2}$ into $y^1_{\alpha-1_1+1_2} + c_1 t y^1_\alpha + \dots + c_{\alpha_2+1} t^{\alpha_2+1} y^1_{(\alpha_1+\alpha_2, 0, \alpha_3, \dots, \alpha_m)}$ for some $c_1, \dots \in \mathbf{R}$ with $c_1 \ne 0$, see Lemma 3.1 (iv).

Then using the invariance of C with respect to χ_t, we get

$$\left\langle C((y^1_{\alpha-1_1+1_2}+c_1ty^1_\alpha+\dots)dx^\mu+dx^\mu,x^m+c)_\theta,\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$
$$=\left\langle C(y^1_{\alpha-1_1+1_2}dx^\mu+dx^\mu,x^m+c)_\theta,\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$

for any $t\in\mathbf{R}$. Then since $\left\langle C(Ldx^\mu+dx^\mu,x^m+c)_\theta,\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$ depends linearly on L, we get $\left\langle C(y^1_\alpha dx^\mu+dx^\mu,x^m+c)_\theta,\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle=0$.

So, C is determined by the collection of values

$$\left\langle C(y^1_{(0,\dots,0,k)}dx^\mu+dx^\mu,x^m+c)_\theta,\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle\in\bigwedge^m T^*_0\mathbf{R}^m$$

for all $c\in\mathbf{R}$ and $k=0,1,\dots,r$.

Consequently, C is determined by the collection of (smooth because C is regular) maps $C^{\langle k\rangle}:\mathbf{R}\to\mathbf{R}$ for $k=0,\dots,s$ defined by

$$C^{\langle k\rangle}(c)dx^\mu_{|0}:=\left\langle C(y^1_{(0,\dots,0,k)}dx^\mu+dx^\mu,x^m+c)_\theta,\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle,\quad c\in\mathbf{R}.$$

More precisely, if C' is an another operator in question such that $C^{\langle k\rangle}=(C')^{\langle k\rangle}$ for $k=0,\dots,s$, then $C=C'$.

On the other hand, given a collection of maps $h_l:\mathbf{R}\to\mathbf{R}$ for $l=0,\dots,s$, we have $(\sum_{l=0}^s h_l\cdot E^{(l)})^{\langle k\rangle}=h_k$ for $k=0,1,\dots,s$. Indeed, using the coordinate expression of the Euler map $E(\lambda)$ from [3], we have

$$\begin{aligned}&\left(\sum_{l=0}^s h_l\cdot E^{(l)}\right)^{\langle k\rangle}(c)dx^\mu_{|0}\\&\quad=\sum_{l=0}^s h_l(c)\left\langle E((-1)^l(x^m)^l(y^1_{(0,\dots,0,k)}dx^\mu+dx^\mu))_{|j^{2s}_0(0)},\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle\\&\quad=\sum_{l=1}^s h_l(c)(-1)^l(-1)^k\frac{1}{k!}\frac{\partial^k}{\partial(x^m)^k}\frac{\partial}{\partial y^k_{(0,\dots,0,k)}}((x^m)^ly^1_{(0,\dots,0,k)})_{|j^{2s}_0(0)}dx^\mu_{|0}\\&\quad=h_k(c)dx^\mu_{|0}.\end{aligned}$$

The proof of the theorem is complete.

5. Schema of the proof of Theorem 2.6. The proof of Theorem 2.6 is the following modification of the one of Theorem 2.2.

Because of the invariance of C with respect to the $\mathcal{FM}_{m,n}$-charts, C is determined by the collection of values

$$\langle C(\lambda,g)_\rho, v\rangle \in \bigwedge^m T_0^*\mathbf{R}^m$$

for all $\lambda\in\mathcal{C}^\infty_{J^s\mathbf{R}^{m,n}}(J^s\mathbf{R}^{m,n}, V^*J^s\mathbf{R}^{m,n}\otimes\bigwedge^m T^*\mathbf{R}^m)$, $g\in\mathcal{C}^\infty(\mathbf{R}^m,\mathbf{R})$, $v\in T_0\mathbf{R}^n=V_{(0,0)}\mathbf{R}^{m,n}$ and $\rho\in J_0^{2s}(\mathbf{R}^m,\mathbf{R}^n)=J_0^{2s}(\mathbf{R}^{m,n})$. Quite similarly as in the proof of Theorem 2.2, we can assume that $\rho=\theta=j_0^{2s}(0)$, $g=x^m+c$ and $v=\frac{\partial}{\partial y^1}_{|(0,0)}$.

Further, we can write $\lambda=\sum L_k^\beta((x^i),(y^j_\alpha))\tilde{d}y^k_\beta\otimes dx^\mu$, where L^β_k are real valued maps for $k=1,\dots,n$ and all $\beta\in\mathbf{N}^m$ with $|\beta|\le s$ and where $\tilde{d}h$ denotes the restriction to VJ^sY of the differential dh of $h: J^sY\to\mathbf{R}$.

Because of the π^{2s}_s-locality of C, we may assume that L^β_k are polynomials in $((x^i),(y^j_\alpha))$ of degree $\le q$, where q is an arbitrary positive integer.

Further, quite similarly as in the proof of Theorem 2.2, by the invariance of C with respect to $\psi_\tau=(x^1,\dots,x^m,\frac{1}{\tau^1}y^1,\dots,\frac{1}{\tau^n}y^n)$ for any $(\tau^1,\dots,\tau^n)\in(\mathbf{R}\setminus\{0\})^n$ and the homogeneous function theorem, we can conclude that $\left\langle C(\lambda,x^m+c)_\theta,\frac{\partial}{\partial y^1}_{(0,0)}\right\rangle$ is linear in λ and C is determined by the collection of values

$$\left\langle C(x^\beta\tilde{d}y^1_\alpha\otimes dx^\mu, x^m+c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$

for all $\alpha,\beta\in\mathbf{N}^m$ with $|\alpha|\le s$.

Then by the respective part of proof of Theorem 2.2 with $x^\beta\tilde{d}y^1_\alpha\otimes dx^\mu$ instead of $x^\beta y^1_\alpha dx^\mu+dx^\mu$ and using $\tilde{d}(x^iy^1_\alpha)=x^i\tilde{d}y^1\alpha$ (being the consequence of $dh=0$ on VJ^sY for any $h:M\to\mathbf{R}$), we finally conclude that C is determined by the collection of values

$$\left\langle C(\tilde{d}y^1_{(0,\dots,0,k)}\otimes dx^\mu, x^m+c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle \in \bigwedge^m T_0^*\mathbf{R}^m$$

for all $c\in\mathbf{R}$ and $k=0,1,\dots,s$.

Consequently, C is determined by the collection of (smooth) maps $C^{\langle k\rangle}:\mathbf{R}\to\mathbf{R}$ for $k=0,\dots,s$ defined by

$$C^{\langle k\rangle}(c)dx^\mu_{|0} := \left\langle C(\tilde{d}y^1_{(0,\dots,0,k)}\otimes dx^\mu, x^m+c)_\theta, \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle,\quad c\in\mathbf{R}.$$

On the other hand, given a collection of maps $h_l:\mathbf{R}\to\mathbf{R}$ for $l=0,\dots,s$, one can see that $(\sum_{l=0}^s h_l\cdot\mathbf{E}^{(l)})^{\langle k\rangle}=h_k$ for $k=0,1,\dots,s$.

The proof of Theorem 2.6 is complete.

6. Final observations. Let $\mathcal{AB}_{m,n}$ denote the category of all affine bundles $A \to M$ with m-dimensional bases and n-dimensional fibres and their affine bundle isomorphisms onto open images. It is easy to see that in fact we have also deduced the following results:

Theorem 6.1. *Let m,n,s be positive integers. If $m \geq 2$, then any regular, π^{2s}_s-local and $\mathcal{AB}_{m,n}$-natural (i.e. invariant with respect to $\mathcal{AB}_{m,n}$-maps) operator*

$$C : \mathcal{C}^\infty_M\left(J^sA, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M,\mathbf{R}) \to \mathcal{C}^\infty_A\left(J^{2s}A, V^*A \otimes \bigwedge^m T^*M\right)$$

is $C = \sum_{l=0}^{s} h_l \cdot E^{(l)}$ for some (uniquely determined by C) maps $h_l : \mathbf{R} \to \mathbf{R}$, $l = 0,\dots,s$. In other words, the space of all C in question is the free $(s+1)$-dimensional $\mathcal{C}^\infty(\mathbf{R})$-module and the operators $E^{(l)}$ for $l = 0,1,\dots,s$ form the basis in this module.

Theorem 6.2. *Let m,n,s be positive integers. If $m \geq 2$, then any regular, π^{2s}_s-local and $\mathcal{AB}_{m,n}$-natural operator*

$$C : \mathcal{C}^\infty_{J^sA}\left(J^sA, V^*J^sA \otimes \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M,\mathbf{R}) \to \mathcal{C}^\infty_A\left(J^{2s}A, V^*A \otimes \bigwedge^m T^*M\right)$$

is $C = \sum_{l=0}^{s} h_l \cdot \mathbf{E}^{(l)}$ for some (uniquely determined by C) maps $h_l : \mathbf{R} \to \mathbf{R}$, $l = 0,\dots,s$. So, the space of all C in question is also the free $(s+1)$-dimensional $\mathcal{C}^\infty(\mathbf{R})$-module and the operators $\mathbf{E}^{(l)}$ for $l = 0,1,\dots,s$ form the basis in this module.

Proof. Indeed, all $\mathcal{FM}_{m,n}$-maps we used in the previous sections are in fact $\mathcal{AB}_{m,n}$-maps, except $\mathcal{FM}_{m,n}$-charts. But they may be replaced by $\mathcal{AB}_{m,n}$-charts if we study $\mathcal{AB}_{m,n}$-natural operators. □

Theorem 2.2 is not true in the vector bundle situation instead of the fibered manifold one because we have:

Example 6.3. Let $\lambda : J^sH \to \bigwedge^m T^*M$ be an sth order Lagrangian on a vector bundle $H \to M$. Then the derivative $\mathcal{E}\lambda : J^sH \to \bigwedge^m T^*M$ of λ with respect to $\mathcal{E}$, where $\mathcal{E}$ is the Euler (dilatation) vector field on the vector bundle $J^rH \to M$, is also an sth order Lagrangian on $H \to M$. Let $U(\lambda) := E(\mathcal{E}\lambda) : J^{2s}H \to V^*H \otimes \bigwedge^m T^*M$, where E is the Euler operator. Then we have the resulting operator

$$U : \mathcal{C}^\infty_M\left(J^sH, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_H\left(J^{2s}H, V^*H \otimes \bigwedge^m T^*M\right).$$

Of course, U is $\mathcal{VB}_{m,n}$-natural, regular, π^{2s}_s-local and (even) linear but it is not of the form cE, where $\mathcal{VB}_{m,n}$ denotes the category of vector bundles with m-dimensional bases and n-dimensional fibres and their vector bundle isomorphisms onto open images. Suppose, $U = \sum_{l=0}^{s} h_l \cdot E^{(l)}$. Then $U(\lambda) = U(\lambda, 1) = h_0(1)E(\lambda)$, i.e. $U = h_0(1)E$ which is a contradiction.

References

[1] Kolář, I., *Natural operators related with variational calculus*, in: *Differential Geometry and Its Applications (Opava, 1992)*, 461–472, Math. Publ. **1**, Silesian Univ. Opava, Opava, 1993.

[2] Kolář, I., *A geometrical version of the higher order Hamiltonian formalism in fibered manifolds*, J. Geom. Phys. **1** (1984), 127–137.

[3] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

[4] Kolář, I., Vitolo, R., *On the Helmholz operators for Euler morphisms*, Math. Proc. Cambridge Philos. Soc. **135** (2003), 277–290.

[5] Mikulski, W. M., *On regular local operators on smooth maps*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **62**(2) (2015), 69–72.

[6] Mikulski, W. M., *On naturality of the formal Euler operator*, Demonstratio Math. **38** (2005) 235–238.

Jan Kurek
Institute of Mathematics
M. Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin
Poland
e-mail: jan.kurek@mail.umcs.pl

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
ul. Łojasiewicza 6
30-348 Cracow
Poland
e-mail: wlodzimierz.mikulski@im.uj.edu.pl

Received February 22, 2024