doi: 10.17951/a.2023.77.1.1-12

A N N A L E S
U N I V E R S I T A T I S M A R I A E C U R I E - S K Ł O D O W S K A
L U B L I N – P O L O N I A

VOL. LXXVII, NO. 1, 2023 SECTIO A 1–12

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A new hybrid generalization of Fibonacci and Fibonacci–Narayana polynomials

Abstract. The hybrid numbers are generalization of complex, hyperbolic and dual numbers. The hybrinomials are polynomials which generalize hybrid numbers. In this paper, we introduce and study the distance Fibonacci hybrinomials, i.e. hybrinomials with coefficients being distance Fibonacci polynomials.

1. Introduction.

The hybrid numbers were introduced by Özdemir in [13] as a new generalization of complex, hyperbolic and dual numbers.

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},$$

where the coefficients a, b, c, d are real numbers and $\mathbf{i}$, ε, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1 \tag{1}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{2}$$

The addition and subtraction of hybrid numbers are done by adding and subtracting corresponding terms and hence their coefficients. The hybrid

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2010 *Mathematics Subject Classification.* 11B39, 11B37.

Key words and phrases. Fibonacci numbers, recurrence relations, hybrid numbers, hybrinomials.

numbers multiplication is defined using (1) and (2). Note that using the formulas (1) and (2), we can find the product of any two hybrid units. The following Table 1 presents products of $\mathbf{i}$, ε, and $\mathbf{h}$.

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

TABLE 1. The hybrid numbers multiplication.

Using the rules given in Table 1, the multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. Moreover, $(\mathbb{K},+,\cdot)$ is a non-commutative ring.

The Fibonacci numbers F_n are defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial terms $F_0 = 0$, $F_1 = 1$. The Lucas numbers L_n are defined by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$. The Fibonacci–Narayana numbers N_n are defined as follows $N_n = N_{n-1} + N_{n-3}$ for $n \geq 3$ with $N_0 = 0$, $N_1 = 1$, $N_2 = 1$, for details see [10].

For any variable quantity x, the Fibonacci polynomials $F_n(x)$ are defined as $F_n(x) = x \cdot F_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$ with $F_0(x) = 0$, $F_1(x) = 1$. The Lucas polynomials $L_n(x)$ are defined as $L_n(x) = x \cdot L_{n-1}(x) + L_{n-2}(x)$ for $n \geq 2$ with initial terms $L_0(x) = 2$, $L_1(x) = x$. The Fibonacci–Narayana polynomials $N_n(x)$ are defined by the formula $N_n(x) = x \cdot N_{n-1}(x) + N_{n-3}(x)$ for $n \geq 3$ with $N_0(x) = 0$, $N_1(x) = 1$, $N_2(x) = x$.

For $x = 1$ the Fibonacci, Lucas and Fibonacci–Narayana polynomials give the Fibonacci, Lucas and Fibonacci–Narayana numbers, respectively. Properties of Fibonacci, Lucas and Fibonacci–Narayana polynomials can be found in [5, 6, 7, 9, 14, 17, 23, 24], among others. In recent years, many interesting papers investigating the properties of Narayana numbers and Narayana polynomials have been published, see e.g. [8, 12, 15, 16, 18].

Fibonacci hybrid numbers were defined and studied in [19]. In [20], the authors presented some properties of Fibonacci and Lucas hybrid numbers. Fibonacci–Narayana hybrid numbers (with initial conditions 1, 1, 1) were examined in [22].

The nth Fibonacci hybrid number FH_n, the nth Lucas hybrid number LH_n and the nth Fibonacci–Narayana hybrid number are defined as

$$FH_n = F_n + \mathbf{i}F_{n+1} + \varepsilon F_{n+2} + \mathbf{h}F_{n+3}, \tag{3}$$

$$LH_n = L_n + \mathbf{i}L_{n+1} + \varepsilon L_{n+2} + \mathbf{h}L_{n+3}, \tag{4}$$

$$NH_n = N_n + \mathbf{i}N_{n+1} + \varepsilon N_{n+2} + \mathbf{h}N_{n+3}, \tag{5}$$

respectively.

The hybrinomials are polynomials, which are a generalization of hybrid numbers. The term „hybrinomials" was used for the first time in [21], where Fibonacci and Lucas hybrinomials were studied. The Narayana polynomials (with initial conditions 2, 3, 4) and Narayana hybrinomials were considered in [17]. Some generalization of Fibonacci and Lucas hybrinomials was introduced in [1]. The authors defined a class of hybrid polynomials (hybrinomials), which are so-called "r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s".

We recall that for $n \geq 0$ the Fibonacci and Lucas hybrinomials are defined by

$$FH_n(x) = F_n(x) + \mathbf{i}F_{n+1}(x) + \varepsilon F_{n+2}(x) + \mathbf{h}F_{n+3}(x) \tag{6}$$

and

$$LH_n(x) = L_n(x) + \mathbf{i}L_{n+1}(x) + \varepsilon L_{n+2}(x) + \mathbf{h}L_{n+3}(x), \tag{7}$$

where $F_n(x)$ is the nth Fibonacci polynomial, $L_n(x)$ is the the nth Lucas polynomial and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid units which satisfy (1) and (2).

By analogy, for $n \geq 0$ the Fibonacci–Narayana hybrinomials are defined by

$$NH_n(x) = N_n(x) + \mathbf{i}N_{n+1}(x) + \varepsilon N_{n+2}(x) + \mathbf{h}N_{n+3}(x). \tag{8}$$

Using formulas (6)–(8), for $x = 1$ we obtain the Fibonacci hybrid numbers, the Lucas hybrid numbers and the Fibonacci–Narayana hybrid numbers, respectively.

In the literature we can find many generalizations of Fibonacci and Lucas numbers, see for example the list in [2]. The authors generalized the definition of the Fibonacci numbers by changing the initial conditions, changing the recurrence relation or changing distance between terms of a sequence. One of the generalizations in the distance sense was introduced in [11] as follows.

Let $k \geq 2$, $n \geq 0$ be integers. The generalized Fibonacci numbers $F(k, n)$ and generalized Lucas numbers $L(k, n)$ were defined as

$$\begin{aligned} &F(k,n) = n+1 \quad \text{for } n = 0, 1, \ldots, k-1, \\ &F(k,n) = F(k,n-1) + F(k,n-k) \quad \text{for } n \geq k \end{aligned}$$

and

$$\begin{aligned} &L(k,n) = n+1 \quad \text{for } n = 0, 1, \ldots, 2k-1, \\ &L(k,n) = L(k,n-1) + L(k,n-k) \quad \text{for } n \geq 2k. \end{aligned}$$

Table 2 presents initial words of generalized Fibonacci numbers and generalized Lucas numbers for special cases of n and k.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
$F(2,n)$	1	2	3	5	8	13	21	34	55	89	144
N_n	0	1	1	1	2	3	4	6	9	13	19
$F(3,n)$	1	2	3	4	6	9	13	19	28	41	60
$F(4,n)$	1	2	3	4	5	7	10	14	19	26	36
$F(5,n)$	1	2	3	4	5	6	8	11	15	20	26
L_n	2	1	3	4	7	11	18	29	47	76	123
$L(2,n)$	1	2	3	4	7	11	18	29	47	76	123
$L(3,n)$	1	2	3	4	5	6	10	15	21	31	46
$L(4,n)$	1	2	3	4	5	6	7	8	13	19	26

TABLE 2. The values of $F(k,n)$, $L(k,n)$, F_n, N_n and L_n.

Note that for $n \geq 0$ we have $F(2,n) = F_{n+2}$ and for $n \geq 2$ holds $L(2,n) = L_n$. Moreover, $F(3,n) = N_{n+3}$.

In [22], the authors defined $F(k,n)$-Fibonacci hybrid numbers FH_n^k and $L(k,n)$-Lucas hybrid numbers LH_n^k as follows. Let $n \geq 0$, $k \geq 2$ be integers. Then

$$FH_n^k = F(k,n) + \mathbf{i}F(k,n+1) + \varepsilon F(k,n+2) + \mathbf{h}F(k,n+3), \tag{9}$$

$$LH_n^k = L(k,n) + \mathbf{i}L(k,n+1) + \varepsilon L(k,n+2) + \mathbf{h}L(k,n+3). \tag{10}$$

For $k = 2$ we obtain $FH_n^2 = FH_{n+2}$ and $LH_n^2 = LH_n$. For $k = 3$ we have $FH_n^3 = NH_{n+3}$.

In [3], the authors introduced distance Fibonacci polynomials as a generalization of Fibonacci and Fibonacci–Narayana polynomials. Let $k \geq 2$, $n \geq 0$ be integers. The distance Fibonacci polynomials $f_n(k,x)$ are given by the following recurrence relation

$$f_n(k,x) = xf_{n-1}(k,x) + f_{n-k}(k,x) \quad \text{for } n \geq k \tag{11}$$

with initial conditions $f_n(k,x) = x^n$ for $n = 0, 1, \ldots, k-1$.

Table 3 presents some distance Fibonacci polynomials $f_n(k,x)$ for special values of k and n.

n	0	1	2	3	4	5	6
$f_n(2,x)$	1	x	x^2+1	x^3+2x	x^4+3x^2+1	x^5+4x^3+3x	$x^6+5x^4+6x^2+1$
$f_n(3,x)$	1	x	x^2	x^3+1	x^4+2x	x^5+3x^2	x^6+4x^3+1
$f_n(4,x)$	1	x	x^2	x^3	x^4+1	x^5+2x	x^6+3x^2
$f_n(5,x)$	1	x	x^2	x^3	x^4	x^5+1	x^6+2x

TABLE 3. Distance Fibonacci polynomials $f_n(k,x)$.

Note that $f_n(2,x) = F_{n+1}(x)$ and $f_n(3,x) = N_{n+1}(x)$.

The generalization of Fibonacci numbers and Fibonacci polynomials is the motivation to generalize Fibonacci hybrinomials in terms of distance. Based on the definition of distance Fibonacci polynomials we will define distance Fibonacci hybrinomials in the following way.

For $n \geq 0$ the distance Fibonacci hybrinomials are defined by

$$fH_n^k(x) = f_n(k,x) + \mathbf{i}f_{n+1}(k,x) + \varepsilon f_{n+2}(k,x) + \mathbf{h}f_{n+3}(k,x), \tag{12}$$

where $f_n(k,x)$ is the nth distance Fibonacci polynomial and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid units which satisfy (1) and (2).

In the next section we will present some properties of these hybrinomials.

2. Main results. We start with the recurrence relations for distance Fibonacci hybrinomials.

Theorem 1. *Let $k \geq 2$, $n \geq 0$ be integers. For any variable quantity x, we have*

$$fH_n^k(x) = x \cdot fH_{n-1}^k(x) + fH_{n-k}^k(x) \quad \textit{for } n \geq k$$

with

$$\begin{aligned}
fH_0^k(x) &= f_0(k,x) + \mathbf{i}f_1(k,x) + \varepsilon f_2(k,x) + \mathbf{h}f_3(k,x),\\
fH_1^k(x) &= f_1(k,x) + \mathbf{i}f_2(k,x) + \varepsilon f_3(k,x) + \mathbf{h}f_4(k,x),\\
&\vdots\\
fH_{k-1}^k(x) &= f_{k-1}(k,x) + \mathbf{i}f_k(k,x) + \varepsilon f_{k+1}(k,x) + \mathbf{h}f_{k+2}(k,x).
\end{aligned}$$

Proof. For an integer n, $n \geq k$, using the definition of the distance Fibonacci polynomials, we have

$$\begin{aligned}
fH_n^k(x) &= f_n(k,x) + \mathbf{i}f_{n+1}(k,x) + \varepsilon f_{n+2}(k,x) + \mathbf{h}f_{n+3}(k,x)\\
&= (x \cdot f_{n-1}(k,x) + f_{n-k}(k,x)) + \mathbf{i}(x \cdot f_n(k,x) + f_{n-k+1}(k,x))\\
&\quad + \varepsilon(x \cdot f_{n+1}(k,x) + f_{n-k+2}(k,x)) + \mathbf{h}(x \cdot f_{n+2}(k,x) + f_{n-k+3}(k,x))\\
&= x\,(f_{n-1}(k,x) + \mathbf{i}f_n(k,x) + \varepsilon f_{n+1}(k,x) + \mathbf{h}f_{n+2}(k,x))\\
&\quad + f_{n-k}(k,x) + \mathbf{i}f_{n-k+1}(k,x) + \varepsilon f_{n-k+2}(k,x) + \mathbf{h}f_{n-k+3}(k,x)\\
&= x \cdot fH_{n-1}^k(x) + fH_{n-k}^k(x),
\end{aligned}$$

which ends the proof. □

Theorem 2. *Let $n \geq 0$, $k \geq 2$ be integers. The generating function of the distance Fibonacci hybrinomials sequence $\{fH_n^k(x)\}$ has the following form*

$$g(t)=\frac{fH_0^k(x)+\big(fH_1^k(x)-x\cdot fH_0^k(x)\big)\,t+\cdots+\big(fH_{k-1}^k(x)-x\cdot fH_{k-2}^k(x)\big)\,t^{k-1}}{1-xt-t^k}.$$

Proof. Assume that the generating function of the distance Fibonacci hybrinomials sequence $\{fH_n^k(x)\}$ has the form $g(t) = \sum_{n=0}^{\infty} fH_n^k(x)t^n$. Then

$$\begin{aligned} g(t) = fH_0^k(x) &+ fH_1^k(x)t + fH_2^k(x)t^2 + \cdots + fH_{k-1}^k(x)t^{k-1} \\ &+ fH_k^k(x)t^k + fH_{k+1}^k(x)t^{k+1} + fH_{k+2}^k(x)t^{k+2} + \cdots \end{aligned}$$

Multiplying the above equality on both sides by $-xt$ and then by $-t^k$, we obtain

$$\begin{aligned} -g(t)xt &= -fH_0^k(x)xt - fH_1^k(x)xt^2 - fH_2^k(x)xt^3 - \cdots - fH_{k-1}^k(x)xt^k \\ &\quad - fH_k^k(x)xt^{k+1} - fH_{k+1}^k(x)xt^{k+2} - fH_{k+2}^k(x)xt^{k+3} + \cdots \\ -g(t)t^k &= -fH_0^k(x)t^k - fH_1^k(x)t^{k+1} - fH_2^k(x)t^{k+2} - \cdots \end{aligned}$$

Adding the above three equalities, we get

$$\begin{aligned} g(t)(1-xt-t^k) &= fH_0^k(x) + fH_1^k(x)t + fH_2^k(x)t^2 + \cdots + fH_{k-1}^k(x)t^{k-1} \\ &\quad - fH_0^k(x)xt - fH_1^k(x)xt^2 - fH_2^k(x)xt^3 - \cdots - fH_{k-2}^k(x)xt^{k-1} \\ &= fH_0^k(x) + \Big(fH_1^k(x) - x\cdot fH_0^k(x)\Big)\,t + \cdots \\ &\quad + \Big(fH_{k-1}^k(x) - x\cdot fH_{k-2}^k(x)\Big)\,t^{k-1} \end{aligned}$$

since $fH_n^k(x) = x\cdot fH_{n-1}^k(x) + fH_{n-k}^k(x)$ for $n \geq k$ and the coefficients of t^n for $n \geq k$ are equal to zero. □

As a special case, we obtain the generating function of the Fibonacci hybrinomials, given in [21].

Corollary 3. *Let $n \geq 0$ be an integer. The generating function of the Fibonacci hybrinomials sequence $\{FH_n(x)\}$ has the following form*

$$G(t) = \frac{\mathbf{i} + \varepsilon x + \mathbf{h}(x^2+1) + (1+\varepsilon+\mathbf{h}x)t}{1-xt-t^2}.$$

Proof. For $k=2$ we have

$$g(t) = \frac{fH_0^2(x) + \big(fH_1^2(x) - x\cdot fH_0^2(x)\big)\,t}{1-xt-t^2}.$$

Moreover,

$$
\begin{aligned}
fH_0^2(x) &= f_0(2,x) + \mathbf{i}f_1(2,x) + \varepsilon f_2(2,x) + \mathbf{h}f_3(2,x) \\
&= F_1(x) + \mathbf{i}F_2(x) + \varepsilon F_3(x) + \mathbf{h}F_4(x) \\
&= 1 + \mathbf{i}x + \varepsilon(x^2+1) + \mathbf{h}(x^3+2x)
\end{aligned}
$$

$$
\begin{aligned}
fH_1^2(x) - x \cdot fH_0^2(x) &= f_1(2,x) + \mathbf{i}f_2(2,x) + \varepsilon f_2(2,x) + \mathbf{h}f_4(2,x) \\
&\quad - x\left(f_0(2,x) + \mathbf{i}f_1(2,x) + \varepsilon f_2(2,x) + \mathbf{h}f_3(2,x)\right) \\
&= F_2(x) + \mathbf{i}F_3(x) + \varepsilon F_4(x) + \mathbf{h}F_5(x) \\
&\quad - x\left(F_1(x) + \mathbf{i}F_2(x) + \varepsilon F_3(x) + \mathbf{h}F_4(x)\right) \\
&= x + \mathbf{i}(x^2+1) + \varepsilon(x^3+2x) + \mathbf{h}(x^4+3x^2+1) \\
&\quad - x\big(1 + \mathbf{i}x + \varepsilon(x^2+1) + \mathbf{h}(x^3+2x)\big) \\
&= \mathbf{i} + \varepsilon x + \mathbf{h}(x^2+1).
\end{aligned}
$$

$$
\begin{aligned}
g(t) &= \sum_{n=0}^{\infty} fH_n^2(x)t^n = fH_0^2(x) + fH_1^2(x)t + fH_2^2(x)t^2 + \cdots \\
&= FH_1(x) + FH_2(x)t + FH_3(x)t^2 + \cdots \\
&= \frac{1}{t}\left(-FH_0(x) + FH_0(x)\right) + \frac{1}{t}\big(FH_1(x)t + FH_2(x)t^2 + FH_3(x)t^3 + \cdots\big) \\
&= \frac{-FH_0(x)}{t} + \frac{1}{t}\big(FH_0(x) + FH_1(x)t + FH_2(x)t^2 + FH_3(x)t^3 + \cdots\big) \\
&= \frac{-FH_0(x)}{t} + \frac{1}{t} \cdot G(t),
\end{aligned}
$$

where $G(t)$ denotes the generating function of the Fibonacci hybrinomials sequence $\{FH_n(x)\}$. Then we have

$$
\begin{aligned}
G(t) &= t \cdot g(t) + FH_0(x) \\
&= \frac{1 + \mathbf{i}x + \varepsilon(x^2+1) + \mathbf{h}(x^3+2x) + \big(\mathbf{i} + \varepsilon x + \mathbf{h}(x^2+1)\big)t}{1 - xt - t^2} \cdot t \\
&\quad + \mathbf{i} + \varepsilon x + \mathbf{h}(x^2+1) \\
&= \frac{1 + \mathbf{i}x + \varepsilon(x^2+1) + \mathbf{h}(x^3+2x) + \big(\mathbf{i} + \varepsilon x + \mathbf{h}(x^2+1)\big)t}{1 - xt - t^2} \cdot t \\
&\quad + \frac{\big(\mathbf{i} + \varepsilon x + \mathbf{h}(x^2+1)\big)(1 - xt - t^2)}{1 - xt - t^2}
\end{aligned}
$$

and after calculations the result follows. □

In the same way we can prove the next result.

Corollary 4. *Let $n \geq 0$ be an integer. The generating function of the Fibonacci–Narayana hybrinomials sequence $\{NH_n(x)\}$ has the following form*

$$\gamma(t) = \frac{\mathbf{i} + \varepsilon x + \mathbf{h}x^2 + (1+\mathbf{h})t + (\varepsilon + \mathbf{h}x)t^2}{1 - xt - t^3}.$$

In [3], many properties of distance Fibonacci polynomials were given. We will recall two of them which will be useful in the next theorems.

Theorem 5 ([3]). *Let $k \geq 2$, $n \geq 0$, $n \geq k-2$ be integers. Then*

$$x \sum_{i=0}^{n} f_i(k,x) = \sum_{i=n+2-k}^{n+1} f_i(k,x) - 1.$$

Theorem 6 ([3]). *Let $k \geq 2$, $n \geq 0$ be integers. Then*

$$f_n(k,x) = \sum_{i=0}^{k-1} x^i f_{n-k-i}(k,x) + x^k f_{n-k}(k,x).$$

Theorem 7. *Let $k \geq 2$, $n \geq 0$, $n \geq k-2$ be integers. Then*

$$\begin{aligned} x \sum_{i=0}^{n} fH_i^k(x) &= \sum_{i=n+2-k}^{n+1} fH_i^k(x) \\ &\quad - \left(1 + \mathbf{i}(1+x) + \varepsilon(1+x+x^2) + \mathbf{h}(1+x+x^2+xf_2(k,x))\right). \end{aligned} \tag{13}$$

Proof. For integers $k \geq 2$, $n \geq 0$ we have

$$\begin{aligned} x \sum_{i=0}^{n} fH_i^k(x) &= x\left(fH_0^k(x) + fH_1^k(x) + \cdots + fH_n^k(x)\right) \\ &= x\left(f_0(k,x) + \mathbf{i}f_1(k,x) + \varepsilon f_2(k,x) + \mathbf{h}f_3(k,x)\right) \\ &\quad + x\left(f_1(k,x) + \mathbf{i}f_2(k,x) + \varepsilon f_3(k,x) + \mathbf{h}f_4(k,x)\right) + \cdots \\ &\quad + x\left(f_n(k,x) + \mathbf{i}f_{n+1}(k,x) + \varepsilon f_{n+2}(k,x) + \mathbf{h}f_{n+3}(k,x)\right) \\ &= x\sum_{i=0}^{n} f_i(k,x) + \mathbf{i}x\sum_{i=1}^{n+1} f_i(k,x) + \varepsilon x\sum_{i=2}^{n+2} f_i(k,x) + \mathbf{h}x\sum_{i=3}^{n+3} f_i(k,x) \\ &= x\sum_{i=0}^{n} f_i(k,x) + \mathbf{i}x\sum_{i=0}^{n+1} f_i(k,x) + \varepsilon x\sum_{i=0}^{n+2} f_i(k,x) + \mathbf{h}x\sum_{i=0}^{n+3} f_i(k,x) \\ &\quad - \mathbf{i}xf_0(k,x) - \varepsilon xf_0(k,x) - \varepsilon xf_1(k,x) \\ &\quad - \mathbf{h}xf_0(k,x) - \mathbf{h}xf_1(k,x) - \mathbf{h}xf_2(k,x). \end{aligned}$$

By Theorem 5 we get

$$
\begin{aligned}
x\sum_{i=0}^{n} fH_i^k(x) = &\sum_{i=n+2-k}^{n+1} f_i(k,x) - 1 + \mathbf{i}\left(\sum_{i=n+3-k}^{n+2} f_i(k,x) - 1\right)\\
&+ \varepsilon\left(\sum_{i=n+4-k}^{n+3} f_i(k,x) - 1\right) + \mathbf{h}\left(\sum_{i=n+5-k}^{n+4} f_i(k,x) - 1\right)\\
&- \mathbf{i}x - \varepsilon x - \varepsilon x^2 - \mathbf{h}x - \mathbf{h}x^2 - \mathbf{h}x f_2(k,x)\\
= &\sum_{i=n+2-k}^{n+1} \left(f_i(k,x) + \mathbf{i}f_{i+1}(k,x) + \varepsilon f_{i+2}(k,x) + \mathbf{h}f_{i+3}(k,x)\right)\\
&- 1 - \mathbf{i} - \varepsilon - \mathbf{h} - \mathbf{i}x - \varepsilon x - \varepsilon x^2 - \mathbf{h}x - \mathbf{h}x^2 - \mathbf{h}x f_2(k,x)\\
= &\sum_{i=n+2-k}^{n+1} fH_i^k(x) - \big(1 + \mathbf{i}(1+x) + \varepsilon(1+x+x^2)\\
&+ \mathbf{h}(1+x+x^2+x f_2(k,x))\big),
\end{aligned}
$$

which ends the proof. □

Corollary 8. *Let $n \geq 0$ be an integer. Then*

$$
\begin{aligned}
x\sum_{i=0}^{n} FH_i(x) &= \sum_{i=n}^{n+1} FH_i(x) - \big(1 + \mathbf{i}(1+x) + \varepsilon(1+x+x^2)\\
&\quad + \mathbf{h}(1+x+x^2+x(x^2+1))\big)\\
&= FH_n(x) + FH_{n+1}(x) - \big(1 + \mathbf{i}(1+x) + \varepsilon(1+x+x^2)\\
&\quad + \mathbf{h}(1+2x+x^2+x^3)\big).
\end{aligned}
$$

Corollary 9. *Let $n \geq 1$ be an integer. Then*

$$
\begin{aligned}
x\sum_{i=0}^{n} NH_i(x) &= \sum_{i=n-1}^{n+1} NH_i(x) - \big(1 + \mathbf{i}(1+x) + \varepsilon(1+x+x^2)\\
&\quad + \mathbf{h}(1+x+x^2+x^3)\big)\\
&= NH_{n-1}(x) + NH_n(x) + NH_{n+1}(x) - \big(1 + \mathbf{i}(1+x) + \varepsilon(1+x+x^2)\\
&\quad + \mathbf{h}(1+x+x^2+x^3)\big).
\end{aligned}
$$

Theorem 10. *Let $k \geq 2$, $n \geq 0$ be integers. Then*

$$
fH_n^k(x) = \sum_{i=0}^{k-1} x^i fH_{n-k-i}^k(x) + x^k fH_{n-k}^k(x).
$$

Proof. By Theorem 6 we get

$$
\begin{aligned}
fH_n^k(x) &= f_n(k,x) + \mathbf{i} f_{n+1}(k,x) + \varepsilon f_{n+2}(k,x) + \mathbf{h} f_{n+3}(k,x) \\
&= \sum_{i=0}^{k-1} x^i f_{n-k-i}(k,x) + x^k f_{n-k}(k,x) \\
&\quad + \mathbf{i}\left(\sum_{i=0}^{k-1} x^i f_{n+1-k-i}(k,x) + x^k f_{n+1-k}(k,x)\right) \\
&\quad + \varepsilon\left(\sum_{i=0}^{k-1} x^i f_{n+2-k-i}(k,x) + x^k f_{n+2-k}(k,x)\right) \\
&\quad + \mathbf{h}\left(\sum_{i=0}^{k-1} x^i f_{n+3-k-i}(k,x) + x^k f_{n+3-k}(k,x)\right) \\
&= \sum_{i=0}^{k-1} x^i \left(f_{n-k-i}(k,x) + \mathbf{i} f_{n-k-i+1}(k,x) + \varepsilon f_{n-k-i+2}(k,x) + \mathbf{h} f_{n-k-i+3}(k,x)\right) \\
&\quad + x^k \left(f_{n-k}(k,x) + \mathbf{i} f_{n-k+1}(k,x) + \varepsilon f_{n-k+2}(k,x) + \mathbf{h} f_{n-k+3}(k,x)\right) \\
&= \sum_{i=0}^{k-1} x^i fH_{n-k-i}^k(x) + x^k fH_{n-k}^k(x),
\end{aligned}
$$

which ends the proof. □

Concluding Remarks. The recurrences defining the generalized Fibonacci numbers $F(k,n)$ and the generalized Lucas numbers $L(k,n)$ are of the kth order. Hence, it is difficult to obtain the Binet formula for these sequences for any integer k. Applying some graph interpretation of distance Fibonacci polynomials $f_n(k,x)$, the authors of [3] derived the direct formula for $f_n(k,x)$. For integers $k \geq 2$, $n \geq 0$, $x \geq 1$ the explicit closed form expression for the distance Fibonacci polynomial is given by the following formula

$$
f_n(k,x) = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n-(k-1)j}{j} x^{n-kj}.
$$

Using this formula, one can give the direct formula for the nth distance Fibonacci hybrinomial, but, importantly, only for integer x. It would be useful to find the Binet formula for any x. Then some new identities, namely Catalan's, Cassini's, d'Ocagne's and Vajda's for the distance Fibonacci hybrinomials could be found.

In [4], the authors gave a new generalization of Lucas polynomials in the distance sense. Moreover, $L(k,n)$-Lucas hybrid numbers LH_n^k were investigated in [22]. Our results obtained for distance Fibonacci hybrinomials may be a contribution to considerations about properties of distance Lucas hybrinomials.

Acknowledgments. The authors would like to thank the referee for helpful valuable suggestions which resulted in improvements to this paper.

Declarations. Conflict of interest: The authors declare that they have no conflict of interest. Funding: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

References

[1] Ait-Amrane, N. R., Belbachir, H., Tan, E., *On generalized Fibonacci and Lucas hybrid polynomials*, Turkish J. Math. **46** (2022), 2069–2077.

[2] Bednarz, U., Włoch, I., Wołowiec-Musiał, M., *Total graph interpretation of the numbers of the Fibonacci type*, J. Appl. Math. (2015), Article ID 837917, 7 pp.

[3] Bednarz, U., Wołowiec-Musiał, M., *Distance Fibonacci Polynomials*, Symmetry **12** (9) (2020), 1540, 14 pp.

[4] Bednarz, U., Wołowiec-Musiał, M., *Distance Fibonacci Polynomials - part II*, Symmetry **13** (9) (2021), 1723, 10 pp.

[5] Bicknell, M., *A primer for the Fibonacci numbers VII*, Fibonacci Quart. **8** (4) (1970), 407–420.

[6] Bicknell, M., Hoggatt, V. E. Jr., *Roots of Fibonacci polynomials*, Fibonacci Quart. **11** (5) (1973), 271–274.

[7] Catarino, P., *The $h(x)$-Fibonacci Quaternion Polynomials: Some Combinatorial Properties*, Adv. Appl. Clifford Algebr. **26** (1) (2016), 71–79.

[8] Chen, W. Y. C., Wang, L. X. W., Yang, A. L. B., *Schur positivity and the q-log-convexity of the Narayana polynomials*, J. Algebraic Combin. **32** (3) (2010), 303–338.

[9] Horzum, T., Kocer, E. G., *On some properties of Horadam polynomials*, Int. Math. Forum **4** (25–28) (2009), 1243–1252.

[10] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York–Toronto, 2001.

[11] Kwaśnik, M., Włoch, I., *The total number of generalized stable sets and kernels of graphs*, Ars Combin. **55** (2000), 139–146.

[12] Mansour, T., Sun, Y., *Identities involving Narayana polynomials and Catalan numbers*, Discrete Math. **309** (12) (2009), 4079–4088.

[13] Özdemir, M., *Introduction to Hybrid Numbers*, Adv. Appl. Clifford Algebr. **28** (1) (2018), Paper No. 11, 32 pp.

[14] Özkan, E., Altun, İ., *Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials*, Comm. Algebra **47** (10) (2019), 4020–4030.

[15] Özkan, E., Kuloğlu, B., *On the new Narayana polynomials, the Gauss Narayana numbers and their polynomials*, Asian-Eur. J. Math. **14** (6) (2021), Paper No. 2150100, 16 pp.

[16] Özkan, E., Kuloğlu, B., Peters, J. F., *k-Narayana sequence self-similarity. Flip graph views of k-Narayana self-similarity*, Chaos Solitons Fractals **153** (2) (2021), Paper No. 111473, 11 pp.

[17] Petroudi, S. H. J., Pirouz, M., Özkoç Öztürk, A., *The Narayana polynomial and Narayana hybrinomial sequences*, Konuralp J. Math. **9** (1) (2021), 90–99.

[18] Sulanke, R. A., *Counting Lattice Paths by Narayana Polynomials*, Electron. J. Combin. **7** (2000), Research Paper 40, 9 pp.

[19] Szynal-Liana, A., *The Horadam hybrid numbers*, Discuss. Math. Gen. Algebra Appl. **38** (1) (2018), 91–98.

[20] Szynal-Liana, A., Włoch, I., *The Fibonacci hybrid numbers*, Util. Math. **110** (2019), 3–10.

[21] Szynal-Liana, A., Włoch, I., *Introduction to Fibonacci and Lucas hybrinomials*, Complex Var. Elliptic Equ. **65** (10) (2020), 1736–1747.

[22] Szynal-Liana, A., Włoch, I., *On special spacelike hybrid numbers*, Mathematics **8** (10) (2020), 1671, 10 pp.

[23] Webb, W. A., Parberry, E. A., *Divisibility properties of Fibonacci polynomials*, Fibonacci Quart. **7** (5) (1969), 457–463.

[24] Yuan, Y., Zhang, W., *Some identities involving the Fibonacci polynomials*, Fibonacci Quart. **40** (4) (2002), 314–318.

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Received March 9, 2023