doi: 10.17951/a.2022.76.1.31-46

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXVI, NO. 1, 2022 SECTIO A 31–46

JAN KUREK and WŁODZIMIERZ M. MIKULSKI

The twisted gauge-natural bilinear brackets on couples of linear vector fields and linear p-forms

Abstract. We completely describe all gauge-natural operators C which send linear $(p+2)$-forms H on vector bundles E (with sufficiently large dimensional bases) into $\mathbf{R}$-bilinear operators C_H transforming pairs $(X_1 \oplus \omega_1, X_2 \oplus \omega_2)$ of couples of linear vector fields and linear p-forms on E into couples $C_H(X_1 \oplus \omega_1, X_2 \oplus \omega_2)$ of linear vector fields and linear p-forms on E. Further, we extract all C (as above) such that C_0 is the restriction of the well-known Courant bracket and C_H satisfies the Jacobi identity in Leibniz form for all closed linear $(p+2)$-forms H.

1. Introduction. All manifolds considered in the paper are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class $\mathcal{C}^\infty$). Maps between manifolds are assumed to be $\mathcal{C}^\infty$.

A vector field X on a vector bundle E is called linear if $\mathcal{L}_L X = 0$, where $\mathcal{L}$ is the Lie derivative and L is the Euler vector field. A p-form ω on a vector bundle E is called linear if $\mathcal{L}_L\omega = \omega$. Let $\Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ denote the space of couples $X \oplus \omega$ of linear vector fields X and linear p-forms ω on E.

2010 *Mathematics Subject Classification.* 53A55, 53A45, 53A99.

Key words and phrases. Natural operator, linear vector field, linear p-form, Jacobi identity in Leibniz form.

Let $\mathcal{VB}_{m,n}$ be the category of n-rank vector bundles with m-dimensional bases and their vector bundle isomorphism onto images. A $\mathcal{VB}_{m,n}$-gauge-natural operator

$$C : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\times\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big),\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\Big)$$

sending linear $(p+2)$-forms $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$ on $\mathcal{VB}_{m,n}$-objects E into $\mathbf{R}$-bilinear operators

$$C_H : \Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)\times\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)\to\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)$$

is a $\mathcal{VB}_{m,n}$-invariant family of regular operators (functions)

$$\begin{aligned} C &: \Gamma^l_E\Big(\bigwedge^{p+2} T^*E\Big) \\ &\to Lin_2\Big(\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)\times\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big),\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)\Big)\end{aligned}$$

for all $\mathcal{VB}_{m,n}$-objects E, where $Lin_2(U\times V, W)$ denotes the vector space of all bilinear (over $\mathbf{R}$) functions $U\times V\to W$ for any real vector spaces U, V, W.

The first main result of the article is the following theorem.

Theorem 1.1. *Let m, $p\geq 1$ and $n\geq 1$ be fixed integers such that $m\geq p+2$. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\times\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big),\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\Big)$$

is of the form

$$\begin{aligned} C_H(\rho^1,\rho^2) = a[X^1,X^2]\oplus\{&b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 \\ &+ b_4 d i_{X^2}\omega^1 + b_5\mathcal{L}_{X^1} d i_L\omega^2 + b_6\mathcal{L}_{X^2} d i_L\omega^1 \\ &+ c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH + c_3 i_L i_{X^2} d i_{X^1} H \\ &+ c_4 i_L i_{X^1} d i_{X^2} H + c_5 i_L d i_{X^2} i_{X^1} H\}\end{aligned} \tag{1}$$

for arbitrary (uniquely determined by C) reals $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$, where $\rho^i = X^i\oplus\omega^i\in\Gamma^l_E(TE\oplus\bigwedge^p T^*E)$, $H\in\Gamma^l_E(\bigwedge^{p+2}T^*E)$, and where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

A $\mathcal{VB}_{m,n}$-gauge-natural operator C as above satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms if

$$C_H(\rho^1, C_H(\rho^2,\rho^3)) = C_H(C_H(\rho^1,\rho^2),\rho^3) + C_H(\rho^2, C_H(\rho^1,\rho^3)) \tag{2}$$

for all closed linear $(p+2)$-forms $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$ and all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$-objects E.

For example, the twisted Dorfman–Courant bracket given by

$$(3)\quad [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_H := [X^1, X^2] \oplus \{\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1 + i_{X^1}i_{X^2}H\}$$

is a gauge-natural operator in question satisfying the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms.

The second main result of the article is the following theorem.

Theorem 1.2. *If additionally $m \geq p+3$, then any gauge-natural operator C as above satisfying the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms and the initial condition $C_0 = [[-,-]]_0$ satisfies the equality*

$$(4)\qquad C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_{cH}$$

for any closed linear $(p+2)$-form $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$, where $[[-,-]]_H$ is the (above) twisted (H-twisted) Dorfman–Courant bracket and c is an arbitrary (uniquely determined by C) real number.

Theorems 1.1 and 1.2 for $p = 1$ are proved in [4].

From now on, let $\mathbf{R}^{m,n}$ be the trivial vector bundle over $\mathbf{R}^m$ with the standard fibre $\mathbf{R}^n$ and let $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$.

2. The gauge-natural bilinear brackets on couples of linear vector fields and linear p-forms. Let m, n, p be positive integers.

Let $E = (E \to M)$ be a vector bundle from $\mathcal{VB}_{m,n}$.

Applying the tangent and the cotangent functors, we obtain double vector bundles $(TE; E, TM; M)$ and $(T^*E; E, E^*; M)$.

A vector field X on E is called linear if it is a vector bundle map $X : E \to TE$ between $E \to M$ and $TE \to TM$.

Equivalently, a vector field X on E is linear if it has an expression

$$X = \sum_{i=1}^m a^i(x^1, \dots, x^m)\frac{\partial}{\partial x^i} + \sum_{j,k=1}^n b^k_j(x^1, \dots, x^m)y^j\frac{\partial}{\partial y^k}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E.

Equivalently, a vector field X on E is linear iff $\mathcal{L}_L X = 0$, where $\mathcal{L}$ denotes the Lie derivative and L is the Euler vector field on E (in vector bundle coordinates $L = \sum_{j=1}^n y^j \frac{\partial}{\partial y^j}$).

Equivalently, a vector field X on E is linear if $(a_t)_* X = X$ for any $t > 0$, where $a_t : E \to E$ is the fibre-homothety by t.

A p-form ω on E is called linear if the induced vector bundle morphism

$$\omega^\sharp : \oplus^{k-1} TE \to T^*E$$

over the identity on E is also a vector bundle morphism over a map $\oplus^{k-1}TM \to E^*$ on the other side of the double vector bundle.

Equivalently, a p-form ω on E is linear if it has an expression

$$\omega = \sum a_{i_1,\dots,i_p,j}(x)y^j dx^{i_1}\wedge\ldots\wedge dx^{i_p}+\sum b_{i_1,\dots,i_{p-1},j}(x)dy^j\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_{p-1}}$$

in any local vector bundle trivialization $x^1,\dots,x^m,y^1,\dots,y^n$ on E.

Equivalently, a p-form ω on E is linear iff $\mathcal{L}_L\omega = \omega$.

Equivalently, a p-form ω on E is linear iff $(a_{\frac{1}{t}})_*\omega = t\omega$ for any $t > 0$.

We have the following definition being a modification of the general one from [1].

Definition 2.1. A $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$A : \Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\times\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\rightsquigarrow\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)$$

is a $\mathcal{VB}_{m,n}$-invariant family of $\mathbf{R}$-bilinear operators

$$A : \Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)\times\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)\to\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE\oplus\bigwedge^p T^*E)$ is the vector space of linear sections of $TE\oplus\bigwedge^p T^*E$.

Remark 2.2. The $\mathcal{VB}_{m,n}$-invariance of A means that if

$$(X^1\oplus\omega^1, X^2\oplus\omega^2)\in\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)\times\Gamma^l_E\Big(TE\oplus\bigwedge^p T^*E\Big)$$

and

$$(\overline{X}^1\oplus\overline{\omega}^1, \overline{X}^2\oplus\overline{\omega}^2)\in\Gamma^l_{\overline{E}}\Big(T\overline{E}\oplus\bigwedge^p T^*\overline{E}\Big)\times\Gamma^l_{\overline{E}}\Big(T\overline{E}\oplus\bigwedge^p T^*\overline{E}\Big)$$

are φ-related by an $\mathcal{VB}_{m,n}$-map $\varphi : E\to\overline{E}$ (i.e., $\overline{X}^i\circ\varphi = T\varphi\circ X^i$ and $\overline{\omega}^i\circ\varphi = \bigwedge^p T^*\varphi\circ\omega^i$ for $i = 1, 2$), then so are $A(X^1\oplus\omega^1, X^2\oplus\omega^2)$ and $A(\overline{X}^1\oplus\overline{\omega}^1, \overline{X}^2\oplus\overline{\omega}^2)$.

In [2], we proved the following result.

Theorem 2.3. *Let m, $n\geq 1$ and $p\geq 1$ be natural numbers such that $m\geq p+1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A : \Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\times\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)\rightsquigarrow\Gamma^l\Big(T\oplus\bigwedge^p T^*\Big)$$

is of the form

$$\begin{aligned}A(X^1\oplus\omega^1, X^2\oplus\omega^2) = a[X^1,X^2]\oplus\{&b_1\mathcal{L}_{X^1}\omega^2+b_2\mathcal{L}_{X^2}\omega^1+b_3 di_{X^1}\omega^2\\ &+b_4 di_{X^2}\omega^1+b_5\mathcal{L}_{X^1}di_L\omega^2+b_6\mathcal{L}_{X^2}di_L\omega^1\}\end{aligned}\tag{5}$$

for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$.

3. The twisted gauge-natural bilinear brackets on couples of linear vector fields and linear p-forms.

Definition 3.1. A $\mathcal{VB}_{m,n}$-gauge-natural operator

$$C : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(T\oplus\bigwedge^{p} T^*\Big)\times\Gamma^l\Big(T\oplus\bigwedge^{p} T^*\Big), \Gamma^l\Big(T\oplus\bigwedge^{p} T^*\Big)\Big)$$

sending linear $(p+2)$-forms $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$ on $\mathcal{VB}_{m,n}$-objects E into $\mathbf{R}$-bilinear operators

$$C_H : \Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big)\times\Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big)\to\Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big)$$

is a $\mathcal{VB}_{m,n}$-invariant family of regular operators (functions)

$$C : \Gamma^l_E\Big(\bigwedge^{p+2} T^*E\Big)$$
$$\to Lin_2\Big(\Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big)\times\Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big), \Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big)\Big)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $Lin_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbf{R}$) functions $U \times V \to W$ for any real vector spaces U, V, W.

Remark 3.2. The invariance of C means that if $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$ and $\tilde{H} \in \Gamma^l_{\tilde{E}}(\bigwedge^{p+2} T^*\tilde{E})$ are φ-related and

$$(X^1\oplus\omega^1, X^2\oplus\omega^2)\in\Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big)\times\Gamma^l_E\Big(TE\oplus\bigwedge^{p} T^*E\Big)$$

and

$$(\tilde{X}^1\oplus\tilde{\omega}^1, \tilde{X}^2\oplus\tilde{\omega}^2)\in\Gamma^l_{\tilde{E}}\Big(T\tilde{E}\oplus\bigwedge^{p} T^*\tilde{E}\Big)\times\Gamma^l_{\tilde{E}}\Big(T\tilde{E}\oplus\bigwedge^{p} T^*\tilde{E}\Big)$$

are also φ-related by a $\mathcal{VB}_{m,n}$-map $\varphi : E \to \tilde{E}$, then so are $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $C_{\tilde{H}}(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$.

The regularity of C means that C transforms smoothly parametrized families $(H_t, X^1_t\oplus\omega^1_t, X^2_t\oplus\omega^2_t)$ into smoothly parametrized families $C_{H_t}(X^1_t\oplus\omega^1_t, X^2_t \oplus \omega^2_t)$.

Example 3.3. The twisted Dorfman–Courant bracket

$$[[X^1\oplus\omega^1, X^2\oplus\omega^2]]_H := [X^1, X^2]\oplus\{\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1 + i_{X^1}i_{X^2}H\} \quad (6)$$

is a gauge natural operator in the sense of Definition 3.1.

Remark 3.4. Quite similarly, one can introduce the concepts of $\mathcal{VB}_{m,n}$-gauge-natural operators

$$\Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l(T))\,,$$
$$\Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l\Big(\bigwedge^{p} T^*\Big)\Big)\,,$$
$$\Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l(T)\times\Gamma^l\Big(\bigwedge^{p} T^*\Big),\Gamma^l(T)\Big)\,,$$
$$\vdots$$
$$\Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(\bigwedge^{p} T^*\Big)\times\Gamma^l\Big(\bigwedge^{p} T^*\Big),\Gamma^l\Big(\bigwedge^{p} T^*\Big)\Big)\,.$$

For example, a $\mathcal{VB}_{m,n}$-gauge-natural operator

$$\Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l(T)\times\Gamma^l\Big(\bigwedge^{p} T^*\Big),\Gamma^l(T)\Big)$$

is a $\mathcal{VB}_{m,n}$-invariant family of regular operators (functions)

$$\Gamma^l_E\Big(\bigwedge^{p+2} T^*E\Big) \to Lin_2\Big(\Gamma^l_E(TE)\times\Gamma^l_E\Big(\bigwedge^{p} T^*E\Big),\Gamma^l_E(TE)\Big)$$

for all $\mathcal{VB}_{m,n}$-objects E.

Lemma 3.5. *Any $\mathcal{VB}_{m,n}$-gauge-natural operator C in the sense of Definition 3.1 can be considered (in the obvious way) as the system $C = (C^1, C^2, \dots, C^8)$ of $\mathcal{VB}_{m,n}$-gauge natural operators*

$$C^1 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l(T))\,,$$
$$C^2 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l\Big(\bigwedge^{p} T^*\Big)\Big)\,,$$
$$\vdots$$
$$C^8 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(\bigwedge^{p} T^*\Big)\times\Gamma^l\Big(\bigwedge^{p} T^*\Big),\Gamma^l\Big(\bigwedge^{p} T^*\Big)\Big)\,.$$

Proof. The lemma is obvious. □

We prove the following theorem corresponding to Theorem 1.1.

Theorem 3.6. *Let m, $p \geq 1$ and $n \geq 1$ be fixed integers such that $m \geq p+2$. Any $\mathcal{VB}_{m,n}$-gauge-natural operator C in the sense of Definition 3.1 is of the* *form*

$$
\begin{aligned}
C_H(\rho^1,\rho^2) = a[X^1,X^2] \oplus \{& b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 \\
&+ b_4 d i_{X^2}\omega^1 + b_5\mathcal{L}_{X^1} d i_L\omega^2 + b_6\mathcal{L}_{X^2} d i_L\omega^1 \\
&+ c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH + c_3 i_L i_{X^2} d i_{X^1} H \\
&+ c_4 i_L i_{X^1} d i_{X^2} H + c_5 i_L d i_{X^2} i_{X^1} H\}
\end{aligned} \tag{7}
$$

for arbitrary (uniquely determined by C*) reals* $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$*, where* $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$*,* $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$*.*

Proof. For $p = 1$, our theorem is the main result of [4]. So, Theorem 3.6 for $p = 1$ holds. So we may assume that $p \geq 2$.

Consider a $\mathcal{VB}_{m,n}$-gauge-natural operator C in the sense of Definition 3.1. We can easily see that C_0 is a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator in the sense of Definition 2.1. Hence, replacing C by $C - C_0$ and using Theorem 2.3, we may assume that

$$C_0 = 0\,.$$

So, because of Lemma 3.5, our theorem is an immediate consequence of Lemmas 3.7–3.14, below. □

Lemma 3.7. *Let* m, n, p *be fixed positive integers. Any* $\mathcal{VB}_{m,n}$*-gauge-natural operator*

$$C^1 : \Gamma^l\left(\bigwedge^{p+2} T^*\right) \rightsquigarrow Lin_2(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l(T))$$

such that $C^1_0 = 0$ *is* 0*.*

Proof. Using the invariance of C^1 with respect to the fiber homotheties, we get $C^1_{tH}(X, X_1) = C^1_H(X, X_1)$ for any linear vector fields X and X_1 and any linear $(p+2)$-form H on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Putting $t \to 0$, we get $C^1_H(X, X_1) = C^1_0(X, X_1)$. Then (by $C^1_0 = 0$) $C^1_H(X, X_1) = 0$. So, $C^1 = 0$. □

Lemma 3.8. *Let* m*,* $p \geq 2$ *and* $n \geq 1$ *be fixed integers such that* $m \geq p+2$*. Any* $\mathcal{VB}_{m,n}$*-gauge-natural operator*

$$C^2 : \Gamma^l\left(\bigwedge^{p+2} T^*\right) \rightsquigarrow Lin_2\left(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l\left(\bigwedge^{p} T^*\right)\right)$$

such that $C^2_0 = 0$ *is of the form*

$$
\begin{aligned}
C^2_H(X^1,X^2) = {}& c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH + c_3 i_L i_{X^2} d i_{X^1} H \\
&+ c_4 i_L i_{X^1} d i_{X^2} H + c_5 i_L d i_{X^2} i_{X^1} H
\end{aligned} \tag{8}
$$

for arbitrary (uniquely determined by C^2*) reals* c_1, c_2, c_3, c_4, c_5*, where* $X^1, X^2 \in \Gamma^l_E(TE)$ *and* $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$*.*

Proof. Consider arbitrary linear $(p+2)$-forms H and $\tilde{H}$ and linear vector fields X, $\tilde{X}$, X_1 and $\tilde{X}_1$ on $E = \mathbf{R}^{m,n}$.

By the non-linear Peetre theorem (Theorem 19.10 (for $f = 0$) in [1]), there is a positive integer r (independent of H, $\tilde{H}$, X, $\tilde{X}$, X_1, $\tilde{X}_1$) such that the conditions

$$j_0^r(H) = j_0^r(\tilde{H})\,, j_0^r(X) = j_0^r(\tilde{X})\,, j_0^r(X_1) = j_0^r(\tilde{X}_1) \quad (0 \in \mathbf{R}^m)$$

imply

$$j_0^0(C^2_{tH}(tX, tX_1)) = j_0^0(C^2_{t\tilde{H}}(t\tilde{X}, t\tilde{X}_1)) \quad (0 \in \mathbf{R}^m)$$

for a sufficiently small real number $t > 0$ (depending on $H, \tilde{H}, X, \tilde{X}, X_1, \tilde{X}_1$).

Further, using the invariance of C^2 with respect to the fiber homotheties, we get

$$C^2_{tH}(X, X_1) = tC^2_H(X, X_1) \tag{9}$$

for all $t > 0$. (Then $C^2_{tH}(tX, tX_1) = t^3 C^2_H(X, X_1)$ for all $t > 0$.)

Then the conditions

$$j_0^r(H) = j_0^r(\tilde{H})\,, j_0^r(X) = j_0^r(\tilde{X})\,, j_0^r(X_1) = j_0^r(\tilde{X}_1)$$

imply

$$C^2_H(X, X_1)_{|0} = C^2_{\tilde{H}}(\tilde{X}, \tilde{X}_1)_{|0} \quad (0 \in \mathbf{R}^m)\,.$$

Consequently, C^2 is of finite order r. Then $C^2_H(X, X_1)$ is linear in H because of (9) and the homogeneous function theorem.

It is obvious that C^2 is determined by the values

$$i_{X_3} \dots i_{X_{p+2}} C^2_H(X_1, X_2)_{|u} \in \mathbf{R} \tag{10}$$

for all points $u \in \mathbf{R}^{m,n}_0$, all vectors $X_3, \dots, X_{p+2} \in T_u\mathbf{R}^{m,n}$, all linear vector fields X_1 and X_2 and all linear $(p+2)$-forms H on $\mathbf{R}^{m,n}$, where i is the insertion derivative.

Using the 3-linearity of C^2, we can assume that the underlined vector field $\underline{X}_2$ of X_2 is of the form $\underline{X}_2 = fY$ for some "constant" vector field Y on $\mathbf{R}^m$ and some $f : \mathbf{R}^m \to \mathbf{R}$. We can also assume that $u \neq 0$ and

$$T\pi \circ X_{1|u}, Y_{|0}, T\pi(X_3), \dots, T\pi(X_{p+2})$$

are linearly independent (here we use $m \geq p+2$), where π is the bundle projection of $E = \mathbf{R}^{m,n}$. Then, using the $\mathcal{VB}_{m,n}$-invariance of C^2, the 3-linearity of C^2 and the vector bundle version of the Frobenius theorem, we can write

$$u = e_1 = (1, 0, \dots, 0) \in \mathbf{R}^n = \mathbf{R}^{m,n}_0\,, \tag{11}$$

$$H = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_{p+2}} \text{ or } H = x^\alpha dy^k \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p+1}}\,,$$

$$X_1 = \frac{\partial}{\partial x^1}\,, \tag{12}$$

$$(X_2 = x^\beta \frac{\partial}{\partial x^2} \text{ or } X_2 = x^\beta y^k \frac{\partial}{\partial y^l})$$

and

(13)
$$X_3 = \frac{\partial}{\partial x^3}_{|u}, \dots, X_{p+2} = \frac{\partial}{\partial x^{p+2}}_{|u},$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$ are m-tuples of non-negative integers, $i_1, \dots, i_{p+2}$ are integers with $1 \le i_1 < i_2 < \dots < i_{p+2} \le m$, $j_1, \dots, j_{p+1}$ are integers with $1 \le j_1 < j_2 < \dots < j_{p+1} \le m$ and k, l are numbers from $\{1, \dots, n\}$. Let us assume additionally that

(14)
$$i_{X_3} \dots i_{X_{p+2}} C^2_H(X_1, X_2)_{|u} \neq 0 .$$

First we consider the case $H = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_{p+2}}$ and $X_2 = x^\beta \frac{\partial}{\partial x^2}$. Then using the invariance of C^2 with respect to $(\tau_1 x^1, \dots, \tau_m x^m, y^1, \dots, y^n)$, we get the condition

$$\begin{aligned} &\tau_1 \cdot \ldots \cdot \tau_{p+2} \cdot i_{X_3} \dots i_{X_{p+2}} C^2_H(X_1, X_2)_{|u} \\ &\quad = \tau^\alpha \cdot \tau^\beta \cdot \tau_{i_1} \cdot \ldots \cdot \tau_{ip+2} \cdot i_{X_3} \dots i_{X_{p+2}} C^2_H(X_1, X_2)_{|u} . \end{aligned}$$

Then $\alpha = (0)$, $\beta = (0)$, $i_1 = 1$ and ... and $i_{p+2} = p+2$, i.e.,

$$H = y^k dx^1 \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = \frac{\partial}{\partial x^2} .$$

Next, we consider the case $H = x^\alpha dy^k \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p+1}}$ and $X_2 = x^\beta y^k \frac{\partial}{\partial y^l}$. Then (using similar arguments), we get

$$H = dy^k \wedge dx^1 \wedge dx^3 \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = y^k \frac{\partial}{\partial y^l} .$$

Similarly, in the case $H = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_{p+2}}$ and $X_2 = x^\beta y^k \frac{\partial}{\partial y^l}$, we get a contradiction with (14).

Similarly, in the case $H = x^\alpha dy^k \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p+1}}$ and $X_2 = x^\beta \frac{\partial}{\partial x^2}$, we get

(15)
$$(H = x^i dy^k \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = \frac{\partial}{\partial x^2})$$

or

(16)
$$(H = dy^k \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = x^i \frac{\partial}{\partial x^2})$$

for some $i = 1, \dots, p+2$, where $\widehat{a}$ means that a is dropped. If $i = i_o \ge 4$, then using the invariance of C^2 when replacing x^3 by x^{i_o} (and vice-versa), we see that the value (10) for $i = i_o$ is equal (modulo signum) to the value (10) for $i_o = 3$. So, we can assume that $i = 1, 2, 3$.

Consequently, the operator C^2 is determined by the $\mathcal{VB}_{3,n}$-gauge-natural operator

$$\tilde{C}^2 : \Gamma^l\Big(\bigwedge^3 T^*\Big) \rightsquigarrow Lin_2(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l(T^*))$$

given by

$$\tilde{C}^2_{\tilde{H}}(\tilde{X}_1,\tilde{X}_2) := j^* i_{Y_4}\dots i_{Y_{p+2}} C^2_{\tilde{H}\wedge\omega_o}(\tilde{X}_1\times 0,\tilde{X}_2\times 0)\,,$$

$\tilde{X}_1,\tilde{X}_2\in\Gamma^l_{\tilde{E}}(T\tilde{E})$, $\tilde{H}\in\Gamma^l_{\tilde{E}}(\bigwedge^3 T^*\tilde{E})$, where $\tilde{E}$ is a $\mathcal{VB}_{3,n}$-object with base $\tilde{M}$, $x^4,\dots,x^m$ are the usual coordinates on $\mathbf{R}^{m-3}$, $\omega_o := dx^4\wedge\dots\wedge dx^{p+2}$ (since $p\ge 2$, then $m\ge p+2\ge 4$, and then ω_o is well defined), $Y_4 := \frac{\partial}{\partial x^4}$ and ... and $Y_{p+2} := \frac{\partial}{\partial x^{p+2}}$ are considered as linear vector fields on the $\mathcal{VB}_{m,n}$-object $E := \tilde{E}\times\mathbf{R}^{m-3}$ with the base $\tilde{M}\times\mathbf{R}^{m-3}$, $j:\tilde{E}\to E$ is the inclusion $y\to(y,0)$ and j^* denotes the pull-back with respect to j. Of course, $\tilde{C}^2_0 = 0$.

By Theorem 3.6 for $p=1$ (which is proved in [4]), the vector space of all $\mathcal{VB}_{3,n}$-gauge-natural operators

$$\tilde{C} : \Gamma^l\Big(\bigwedge^3 T^*\Big) \rightsquigarrow Lin_2(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l(T^*))$$

with $\tilde{C}_0 = 0$ is of dimension ≤ 5. Consequently, the vector space of all $\mathcal{VB}_{m,n}$-gauge-natural operators

$$C^2 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l\Big(\bigwedge^p T^*\Big)\Big)$$

with $C^2_0 = 0$ is of dimension ≤ 5.

On the other hand, the system of $\mathcal{VB}_{m,n}$-gauge-natural operators

$$D^1,D^2,D^3,D^4,D^5 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l(T)\times\Gamma^l(T),\Gamma^l\Big(\bigwedge^p T^*\Big)\Big)$$

defined by

$$\begin{aligned}
D^1_H(X^1,X^2) &:= i_{X^1} i_{X^2} H\,,\\
D^2_H(X^1,X^2) &:= i_L i_{X^1} i_{X^2} dH\,,\\
D^3_H(X^1,X^2) &:= i_L i_{X^2} d i_{X^1} H\,,\\
D^4_H(X^1,X^2) &:= i_L i_{X^1} d i_{X^2} H\,,\\
D^5_H(X^1,X^2) &:= i_L d i_{X^2} i_{X^1} H
\end{aligned}$$

is linearly independent. Indeed, if

$$a^1 D^1 + a^2 D^2 + a^3 D^3 + a^4 D^4 + a^5 D^5 = 0\,,$$

then (in particular)

$$a^1 i_{X^1} i_{X^2} H + a^2 i_L i_{X^1} i_{X^2} dH + a^3 i_L i_{X^2} d i_{X^1} H + a^4 i_L i_{X^1} d i_{X^2} H + a^5 i_L d i_{X^2} i_{X^1} H = 0$$

for any linear 3-form $\tilde{H}$ and any linear vector fields $\tilde{X}^1, \tilde{X}^2$ on $\mathbf{R}^{3,n}$, where $H = \tilde{H} \wedge \omega_o \in \Gamma^l_{\mathbf{R}^{m,n}}(\bigwedge^{p+2} T^*\mathbf{R}^{m,n})$ and $X^1 = \tilde{X}^1 \times 0$, $X^2 = \tilde{X}^2 \times 0 \in \Gamma^l_{\mathbf{R}^{m,n}}(T\mathbf{R}^{m,n})$ and ω_o is as above. Then

$$(a^1 i_{\tilde{X}^1} i_{\tilde{X}^2} \tilde{H} + a^2 i_L i_{\tilde{X}^1} i_{\tilde{X}^2} d\tilde{H} + a^3 i_L i_{\tilde{X}^2} d i_{\tilde{X}^1} \tilde{H} + a^4 i_L i_{\tilde{X}^1} d i_{\tilde{X}^2} \tilde{H} + a^5 i_L d i_{\tilde{X}^2} i_{\tilde{X}^1} \tilde{H}) \wedge \omega_o = 0$$

for any $\tilde{H}, \tilde{X}^1, \tilde{X}^2$ as above. Then

$$a^1 i_{\tilde{X}^1} i_{\tilde{X}^2} \tilde{H} + a^2 i_L i_{\tilde{X}^1} i_{\tilde{X}^2} d\tilde{H} + a^3 i_L i_{\tilde{X}^2} d i_{\tilde{X}^1} \tilde{H} + a^4 i_L i_{\tilde{X}^1} d i_{\tilde{X}^2} \tilde{H} + a^5 i_L d i_{\tilde{X}^2} i_{\tilde{X}^1} \tilde{H} = 0$$

for any $\tilde{H}, \tilde{X}^1, \tilde{X}^2$ as above. Then

$$a^1 = a^2 = a^3 = a^4 = a^5 = 0\,,$$

because the collection of operators D^1, D^2, D^3, D^4, D^5 is linearly independent for $p = 1$ and $m = 3$ and $n \geq 1$, see [4].

So, the dimension argument ends the proof of our lemma. □

Lemma 3.9. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C^3 : \Gamma^l\left(\bigwedge^{p+2} T^*\right) \rightsquigarrow Lin_2\left(\Gamma^l(T) \times \Gamma^l\left(\bigwedge^{p} T^*\right), \Gamma^l(T)\right)$$

(not necessarily satisfying $C^3_0 = 0$) is 0.

Proof. Using the invariance of C^3 with respect to the fiber homotheties, we get $C^3_{tH}(X, t\omega) = C^3_H(X, \omega)$ for any linear vector field X, any linear p-form ω, any linear $(p+2)$-form H on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Putting $t \to 0$, we get $C^3_H(X, \omega) = C^3_0(X, 0) = 0$. So, $C^3 = 0$. □

Lemma 3.10. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C^4 : \Gamma^l\left(\bigwedge^{p+2} T^*\right) \rightsquigarrow Lin_2\left(\Gamma^l(T) \times \Gamma^l\left(\bigwedge^{p} T^*\right), \Gamma^l\left(\bigwedge^{p} T^*\right)\right)$$

such that $C^4_0 = 0$ is 0.

Proof. Using the invariance of C^4 with respect to the fiber homotheties, we get $C^4_{tH}(X, t\omega) = tC^4_H(X, \omega)$ for any linear vector field X, any linear p-form ω, any linear $(p+2)$-form H on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Then $C^4_{tH}(X, \omega) = C^4_H(X, \omega)$. Putting $t \to 0$, we get $C^4_H(X, \omega) = C^4_0(X, \omega)$. Then (by the assumption $C^4_0 = 0$), $C^4_H(X, \omega) = 0$. So, $C^4 = 0$. □

Lemma 3.11. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C^5 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(\bigwedge^{p} T^*\Big) \times \Gamma^l(T), \Gamma^l(T)\Big)$$

(not necessarily satisfying $C^5_0 = 0$) is 0.

Proof. It is sufficient to apply Lemma 3.9 for $C^3_H(X, \omega) := C^5_H(\omega, X)$. □

Lemma 3.12. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C^6 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(\bigwedge^{p} T^*\Big) \times \Gamma^l(T), \Gamma^l\Big(\bigwedge^{p} T^*\Big)\Big)$$

such that $C^6_0 = 0$ is 0.

Proof. It is sufficient to apply Lemma 3.10 for $C^4_H(X, \omega) := C^6_H(\omega, X)$. □

Lemma 3.13. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C^7 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(\bigwedge^{p} T^*\Big) \times \Gamma^l\Big(\bigwedge^{p} T^*\Big), \Gamma^l(T)\Big)$$

(not necessarily satisfying $C^7_0 = 0$) is 0.

Proof. Using the invariance of C^7 with respect to the fiber homotheties, we get $C^7_{tH}(t\omega, t\omega^1) = C^3_H(\omega, \omega^1)$ for any linear p-forms ω and ω^1, any linear $(p+2)$-form H on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Putting $t \to 0$, we get $C^7_H(\omega, \omega^1) = C^7_0(0, 0) = 0$. So, $C^7 = 0$. □

Lemma 3.14. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C^8 : \Gamma^l\Big(\bigwedge^{p+2} T^*\Big) \rightsquigarrow Lin_2\Big(\Gamma^l\Big(\bigwedge^{p} T^*\Big) \times \Gamma^l\Big(\bigwedge^{p} T^*\Big), \Gamma^l\Big(\bigwedge^{p} T^*\Big)\Big)$$

(not necessarily satisfying $C^8_0 = 0$) is 0.

Proof. Using the invariance of C^8 with respect to the fiber homotheties, we get $C^8_{tH}(t\omega, t\omega_1) = tC^8_H(\omega, \omega_1)$ for any linear p-forms ω and ω_1, any linear $(p+2)$-form H on a $\mathcal{VB}_{m,n}$-object E and any $t > 0$. Then $C^8_{tH}(\omega, t\omega_1) = C^8_H(\omega, \omega_1)$. Putting $t \to 0$, we get $C^8_H(\omega, \omega_1) = C^8_0(\omega, 0) = 0$. So, $C^8 = 0$. □

4. The generalized twisted D-C brackets with the Jacobi identity in Leibniz form.

Definition 4.1. Let C be a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 3.1. We say that C is a generalized twisted Dorfman–Courant bracket if it satisfies the initial condition $C_0 = [[-,-]]_0$, where $[[-,-]]_H$ is the usual twisted (H-twisted) Dorfman–Courant bracket as in Example 3.3.

As an immediate consequence of Theorem 3.6, we get

Lemma 4.2. *Let m, $n \geq 1$ and $p \geq 1$ be natural numbers such that $m \geq p+2$. Any generalized twisted Dorfman–Courant bracket C is of the form*

$$\begin{aligned} C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2] \oplus \{&\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1 + \\ &+ c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH \\ &+ c_3 i_L i_{X^2} d i_{X^1} H + c_4 i_L i_{X^1} d i_{X^2} H \\ &+ c_5 i_L d i_{X^2} i_{X^1} H\} \end{aligned} \tag{17}$$

for any $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$, any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where c_1, c_2, c_3, c_4, c_5 are (uniquely determined by C) real numbers.

Definition 4.3. A $\mathcal{VB}_{m,n}$-gauge-natural operator C in the sense of Definition 3.1 satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms if

$$C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3)) \tag{18}$$

for all closed linear $(p+2)$-forms $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$, all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$-objects E.

Lemma 4.4. *Let C be a generalized twisted Dorfman–Courant bracket of the form* (17). *If C satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms, then*

$$\begin{aligned} &c_3 \mathcal{L}_{X^1} i_L i_{X^3} d i_{X^2} H + c_4 \mathcal{L}_{X^1} i_L i_{X^2} d i_{X^3} H \\ &+ c_5 \mathcal{L}_{X^1} i_L d i_{X^3} i_{X^2} H + c_3 i_L i_{[X^2,X^3]} d i_{X^1} H \\ &+ c_4 i_L i_{X^1} d i_{[X^2,X^3]} H + c_5 i_L d i_{[X^2,X^3]} i_{X^1} H \\ &= -c_3 i_{X^3} d i_L i_{X^2} d i_{X^1} H - c_4 i_{X^3} d i_L i_{X^1} d i_{X^2} H \\ &\quad - c_5 i_{X^3} d i_L d i_{X^2} i_{X^1} H + c_3 i_L i_{X^3} d i_{[X^1,X^2]} H \\ &\quad + c_4 i_L i_{[X^1,X^2]} d i_{X^3} H + c_5 i_L d i_{X^3} d i_{[X^1,X^2]} H \\ &\quad + c_3 \mathcal{L}_{X^2} i_L i_{X^3} d i_{X^1} H + c_4 \mathcal{L}_{X^2} i_L i_{X^1} d i_{X^3} H \\ &\quad + c_5 \mathcal{L}_{X^2} i_L d i_{X^3} i_{X^1} H + c_3 i_L i_{[X^1,X^3]} d i_{X^2} H \\ &\quad + c_4 i_L i_{X^2} d i_{[X^1,X^3]} H + c_5 i_L d i_{[X^1,X^3]} i_{X^2} H \end{aligned} \tag{19}$$

for any linear vector fields X^1, X^2, X^3 and any closed linear $(p+2)$-form H on $\mathbf{R}^{m,n}$.

Proof. For any linear vector fields X^1, X^2, X^3 and any closed linear $(p+2)$-form H on E, we can write

$$C_H(X^1 \oplus 0, C_H(X^2 \oplus 0, X^3 \oplus 0)) = [X^1, [X^2, X^3]] \oplus \Omega\,,$$
$$C_H(C_H(X^1 \oplus 0, X^2 \oplus 0), X^3 \oplus 0) = [[X^1, X^2], X^3] \oplus \Theta\,,$$
$$C_H(X^2 \oplus 0, C_H(X^1 \oplus 0, X^3 \oplus 0)) = [X^2, [X^1, X^3]] \oplus \mathcal{T}\,,$$

where

$$\begin{aligned}\Omega = {} & c_1 \mathcal{L}_{X^1} i_{X^2} i_{X^3} H + c_3 \mathcal{L}_{X^1} i_L i_{X^3} d i_{X^2} H \\ & + c_4 \mathcal{L}_{X^1} i_L i_{X^2} d i_{X^3} H + c_5 \mathcal{L}_{X^1} i_L d i_{X^3} i_{X^2} H \\ & + c_1 i_{X^1} i_{[X^2,X^3]} H + c_3 i_L i_{[X^2,X^3]} d i_{X^1} H \\ & + c_4 i_L i_{X^1} d i_{[X^2,X^3]} H + c_5 i_L d i_{[X^2,X^3]} i_{X^1} H\,,\end{aligned}$$

$$\begin{aligned}\Theta = {} & - c_1 i_{X^3} d i_{X^1} i_{X^2} H - c_3 i_{X^3} d i_L i_{X^2} d i_{X^1} H \\ & - c_4 i_{X^3} d i_L i_{X^1} d i_{X^2} H - c_5 i_{X^3} d i_L d i_{X^2} i_{X^1} H \\ & + c_1 i_{[X^1,X^2]} i_{X^3} H + c_3 i_L i_{X^3} d i_{[X^1,X^2]} H \\ & + c_4 i_L i_{[X^1,X^2]} d i_{X^3} H + c_5 i_L d i_{X^3} d i_{[X^1,X^2]} H\,,\end{aligned}$$

$$\begin{aligned}\mathcal{T} = {} & c_1 \mathcal{L}_{X^2} i_{X^1} i_{X^3} H + c_3 \mathcal{L}_{X^2} i_L i_{X^3} d i_{X^1} H \\ & + c_4 \mathcal{L}_{X^2} i_L i_{X^1} d i_{X^3} H + c_5 \mathcal{L}_{X^2} i_L d i_{X^3} i_{X^1} H \\ & + c_1 i_{X^2} i_{[X^1,X^3]} H + c_3 i_L i_{[X^1,X^3]} d i_{X^2} H \\ & + c_4 i_L i_{X^2} d i_{[X^1,X^3]} H + c_5 i_L d i_{[X^1,X^3]} i_{X^2} H\,.\end{aligned}$$

Since C satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms,

$$\Omega = \Theta + \mathcal{T}.$$

Moreover, the (usual) twisted Dorfman–Courant bracket satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms. Indeed, the (usual) twisted Dorfman–Courant bracket is the restriction of the twisted Courant bracket (which satisfies the Jacobi identity in Leibniz form for closed $(p+2)$-forms, see [3]). So, we have $\Omega = \Theta + \mathcal{T}$ in the case $c_3 = c_4 = c_5 = 0$, too. So, we have (19). □

Lemma 4.5. *Let C be a generalized twisted Dorfman–Courant bracket of the form* (17). *Let $m, n \geq 1$ and $p \geq 1$ be such that $m \geq p+3$. If C satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms, then $c_3 = c_4 = c_5 = 0$.*

Proof. Let $\tilde{\omega}_o := dx^3 \wedge \ldots \wedge dx^{p+1}$ if $p \geq 2$ (then $\tilde{\omega}_o$ is well defined because $m \geq p+1 \geq 3$) and $\tilde{\omega}_o := 1$ if $p = 1$. Putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$, $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = L$ and the closed linear $(p+2)$-form $H := x^1 dx^1 \wedge dx^2 \wedge dy^1 \wedge \tilde{\omega}_o$ into (19), we get

$$
\begin{aligned}
& c_3\cdot 0 + c_4\cdot (y^1dx^1\wedge\tilde\omega_o) + c_5\cdot (y^1dx^1\wedge\tilde\omega_o)\\
& + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0\\
& = -c_3\cdot y^1dx^1\wedge\tilde\omega_o - c_4\cdot 0 - c_5\cdot(-y^1dx^1\wedge\tilde\omega_o)\\
& \quad + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0 + c_3\cdot 0 + c_4\cdot 0\\
& \quad + c_5\cdot 0 + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0\,.
\end{aligned}
$$

Hence $c_3 = -c_4$.

Similarly, let $\tilde\omega_o$ be as above. Putting linear vector fields $X^1 = x^2\frac{\partial}{\partial x^1}$, $X^2 = \frac{\partial}{\partial x^2}$, $X^3 = L$ and the closed linear $(p+2)$-form $H := dx^1\wedge dx^2\wedge dy^1\wedge\tilde\omega_o$ into (19), we get

$$
\begin{aligned}
& c_3\cdot 0 + c_4\cdot y^1dx^2\wedge\tilde\omega_o + c_5\cdot y^1dx^2\wedge\tilde\omega_o\\
& + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0\\
& = -c_3\cdot 0 - c_4\cdot 0 - c_5\cdot(-y^1dx^2\wedge\tilde\omega_o)\\
& \quad + c_3\cdot 0 + c_4\cdot y^1dx^2\wedge\tilde\omega_o + c_5\cdot 0\\
& \quad + c_3\cdot 0 + c_4\cdot(-y^1dx^2\wedge\tilde\omega_o) + c_5\cdot(-y^1dx^2\wedge\tilde\omega_o)\\
& \quad + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0\,.
\end{aligned}
$$

Hence $c_4 = -c_5$.

Now, let $\tilde\omega_o := dx^5\wedge\ldots\wedge dx^{p+3}$ if $p\geq 2$ (then $\tilde\omega_o$ is well defined because $m\geq p+3\geq 5$), and $\tilde\omega_o := 1$ if $p = 1$. Putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$, $X^2 = x^1\frac{\partial}{\partial x^2}$, $X^3 = \frac{\partial}{\partial x^3}$ and the closed linear $(p+2)$-form $H := d(x^2x^4)\wedge dx^3\wedge dy^1\wedge\tilde\omega_o$ (H is well defined because $m\geq p+3\geq 4$) into (19), we get

$$
\begin{aligned}
& c_3\cdot y^1dx^4\wedge\tilde\omega_o + c_4\cdot 0 + c_5\cdot(-y^1dx^4\wedge\tilde\omega_o)\\
& + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0\\
& = -c_3\cdot 0 - c_4\cdot(y^1dx^4\wedge\tilde\omega_o + x^4dy^1\wedge\tilde\omega_o) - c_5\cdot 0\\
& \quad + c_3\cdot y^1dx^4\wedge\tilde\omega_o + c_4\cdot 0 + c_5\cdot 0\\
& \quad + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0\\
& \quad + c_3\cdot 0 + c_4\cdot 0 + c_5\cdot 0\,.
\end{aligned}
$$

Hence $c_4 = 0$.

Consequently, $c_3 = c_4 = c_5 = 0$, as well. □

Thus we have proved

Theorem 4.6. *Let m, $n\geq 1$ and $p\geq 1$ be such that $m\geq p+3$. Any generalized twisted Dorfman–Courant bracket C satisfying the Jacobi identity in Leibniz form for closed linear $(p+2)$-forms is of the form*

$$
\begin{aligned}
C_H(X^1\oplus\omega^1, X^2\oplus\omega^2) = [X^1,X^2]\oplus\{&\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1\\
&+ c_1 i_{X^1}i_{X^2}H + c_2 i_L i_{X^1}i_{X^2}dH\}
\end{aligned}
\tag{20}
$$

for any $H \in \Gamma^l_E(\bigwedge^{p+2} T^*E)$, *any* $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus \bigwedge^p T^*E)$ *and any* $\mathcal{VB}_{m,n}$*-object* E, *where* c_1, c_2 *are (uniquely determined by* C*) real numbers.*

Given $c_1, c_2 \in \mathbf{R}$, *the generalized twisted Dorfman–Courant bracket* C *of the form* (20) *satisfies the Jacobi identity in Leibniz form for closed linear* $(p+2)$*-forms.*

The above theorem implies immediately Theorem 1.2.

References

[1] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

[2] Kurek, J., Mikulski, W. M., *The gauge-natural bilinear brackets on couples of linear vector fields and linear p-forms*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **75**(2) (2021), 73–92.

[3] Mikulski, W. M., *The natural operators similar to the twisted Courant bracket on couples of vector fields and p-forms*, Filomat **44**(12) (2020), 4071–4078.

[4] Mikulski, W. M., *On the gauge-natural operators similar to the twisted Dorfman–Courant bracket*, Opuscula Math **41**(2) (2021), 205–226.

Jan Kurek
Institute of Mathematics
Maria Curie-Skłodowska University
Pl. Marii Curie-Skłodowskiej 1
20-031 Lublin
Poland
e-mail: `kurek@hektor.umcs.lublin.pl`

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
ul. Łojasiewicza 6
30-348 Cracow
Poland
e-mail: `Wlodzimierz.Mikulski@im.uj.edu.pl`

Received February 7, 2022