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Some Hermite–Hadamard type inequalities for the square norm in Hilbert spaces

Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $f:[0,\infty)\to\mathbb{R}$ be convex (concave) on $[0,\infty)$. If $x,y\in H$ with $\operatorname{Re}\langle x,y\rangle\geq 0$, then

$$f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\leq(\geq)\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt$$
$$\leq(\geq)\,\frac{1}{3}\left[f\left(\|x\|^2\right)+f\left[\operatorname{Re}\langle x,y\rangle\right]+f\left(\|y\|^2\right)\right].$$

Some examples for power functions and exponential are also provided.

1. Introduction. Let $\mathbb{R}$ be the set of real numbers and $I\subseteq\mathbb{R}$ be an interval. A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f\left(tx+(1-t)\,t\right)\leq tf\left(x\right)+(1-t)\,f\left(y\right)$$

for all $x,\,y\in I$ and $t\in[0,1]$.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I$ with $a<b$. Then the inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if f is convex, and it is known in the literature as the *Hermite–Hadamard inequality*, after the names of C. Hermite and J. Hadamard

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(see [9]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [7] since it is considered to be one of the most famous inequalities for convex functions due to its usefulness and many applications in various branches of pure and applied mathematics, such as numerical analysis [2], information theory [1], operator theory [5, 6] and others.

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite–Hadamard integral inequality* (see [3, p. 2], [4, p. 2]):

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite–Hadamard inequality (1.1) for the convex function $g(x, y) : [0,1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (1.2) (see [8, p. 106]):

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

In this paper we give some Hermite–Hadamard type inequalities for the integral

$$\int_0^1 f\left(\|(1-t)\,x + ty\|^2\right) dt$$

in the case when f is a convex (concave) function on $[0, \infty)$ and x, y are vectors in the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Some examples for power functions and exponential are also provided.

2. Main results. We have the following Hermite–Hadamard type inequalities:

Theorem 1. *Let* $f:[0,\infty)\to\mathbb{R}$ *be convex (concave) on* $[0,\infty)$. *If* $x,y\in H$ *with* $\operatorname{Re}\langle x,y\rangle\geq 0$, *then*

$$
\begin{aligned}
& f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right) \\
& \leq (\geq) \int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt \\
& \leq (\geq) \frac{1}{3}\left[f\left(\|x\|^2\right)+f\left[\operatorname{Re}\langle x,y\rangle\right]+f\left(\|y\|^2\right)\right].
\end{aligned}
\tag{2.1}
$$

Proof. Observe that, by the properties of norm and inner product, we have for $t\in[0,1]$,

$$\|(1-t)\,x+ty\|^2=(1-t)^2\|x\|^2+2t\,(1-t)\operatorname{Re}\langle x,y\rangle+t^2\|y\|^2.$$

Assume that $f:[0,\infty)\to\mathbb{R}$ is convex on $[0,\infty)$. By using Jensen's integral inequality, we have

$$f\left(\int_0^1\|(1-t)\,x+ty\|^2\,dt\right)\leq\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt. \tag{2.2}$$

Since

$$
\begin{aligned}
&\int_0^1\|(1-t)\,x+ty\|^2\,dt \\
&=\left(\int_0^1(1-t)^2\,dt\right)\|x\|^2+2\left(\int_0^1 t\,(1-t)\,dt\right)\operatorname{Re}\langle x,y\rangle+\left(\int_0^1 t^2dt\right)\|y\|^2 \\
&=\frac{1}{3}\|x\|^2+\frac{1}{3}\operatorname{Re}\langle x,y\rangle+\frac{1}{3}\|y\|^2=\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right),
\end{aligned}
$$

hence by (2.2) we get the first inequality in (2.1).

Consider $\alpha=(1-t)^2$, $\beta=2t\,(1-t)$, $\gamma=t^2\geq 0$ for $t\in[0,1]$. Then

$$\alpha+\beta+\gamma=(1-t)^2+2t\,(1-t)+t^2=(1-t+t)^2=1$$

and by the convexity of f we have

$$
\begin{aligned}
& f\left((1-t)^2\|x\|^2+2t\,(1-t)\operatorname{Re}\langle x,y\rangle+t^2\|y\|^2\right) \\
& \leq (1-t)^2 f\left(\|x\|^2\right)+2t\,(1-t)\,f\left[\operatorname{Re}\langle x,y\rangle\right]+t^2 f\left(\|y\|^2\right)
\end{aligned}
\tag{2.3}
$$

for all $t\in[0,1]$.

If we take the integral in (2.3), we get

$$\begin{aligned}&\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt\\&\leq\int_0^1\left[(1-t)^2 f\left(\|x\|^2\right)+2t\,(1-t)\,f\left[\operatorname{Re}\langle x,y\rangle\right]+t^2 f\left(\|y\|^2\right)\right]dt\\&=\frac{1}{3}\left[f\left(\|x\|^2\right)+f\left[\operatorname{Re}\langle x,y\rangle\right]+f\left(\|y\|^2\right)\right],\end{aligned}$$

which proves the second part of (2.1). □

The first inequality in (2.1) can be improved as follows:

Theorem 2. *Let* $f:[0,\infty)\to\mathbb{R}$ *be convex (concave) on* $[0,\infty)$*. If* $x,y\in H$ *with* $\operatorname{Re}\langle x,y\rangle\geq 0$*, then*

$$\begin{aligned}&f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\&\leq(\geq)\int_0^1 f\left(\left[t^2+(1-t)^2\right]\frac{\|x\|^2+\|y\|^2}{2}+2t\,(1-t)\operatorname{Re}\langle x,y\rangle\right)dt\qquad(2.4)\\&\leq(\geq)\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt.\end{aligned}$$

Proof. By the convexity of f, we also have

$$\begin{aligned}&\frac{1}{2}\left[f\left(\|(1-t)\,x+ty\|^2\right)+f\left(\|(1-t)\,y+tx\|^2\right)\right]\\&\geq f\left(\frac{\|(1-t)\,x+ty\|^2+\|(1-t)\,y+tx\|^2}{2}\right)\\&=f\left(\frac{1}{2}\left[(1-t)^2\|x\|^2+2t\,(1-t)\operatorname{Re}\langle x,y\rangle+t^2\|y\|^2\right.\right.\qquad(2.5)\\&\quad\left.\left.+\,(1-t)^2\|y\|^2+2t\,(1-t)\operatorname{Re}\langle y,x\rangle+t^2\|x\|^2\right]\right)\\&=f\left(\left[t^2+(1-t)^2\right]\left(\frac{\|x\|^2+\|y\|^2}{2}\right)+2t\,(1-t)\operatorname{Re}\langle x,y\rangle\right)\end{aligned}$$

for all $t\in[0,1]$.

By taking the integral in (2.5), we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^1\left[f\left(\|(1-t)\,x+ty\|^2\right)+f\left(\|(1-t)\,y+tx\|^2\right)\right]dt\\
&\geq\int_0^1 f\left(\left[t^2+(1-t)^2\right]\left(\frac{\|x\|^2+\|y\|^2}{2}\right)+2t\,(1-t)\operatorname{Re}\langle x,y\rangle\right)dt\\
&\geq f\left(\int_0^1\left\{\left[t^2+(1-t)^2\right]\left(\frac{\|x\|^2+\|y\|^2}{2}\right)+2t\,(1-t)\operatorname{Re}\langle x,y\rangle\right\}\right)dt\\
&=f\left(\int_0^1\left[t^2+(1-t)^2\right]dt\left(\frac{\|x\|^2+\|y\|^2}{2}\right)\right.\\
&\qquad\left.+2\left(\int_0^1 t\,(1-t)\,dt\right)\operatorname{Re}\langle x,y\rangle\right)dt\\
&=f\left(\frac{\|x\|^2+\|y\|^2+\operatorname{Re}\langle x,y\rangle}{3}\right),
\end{aligned}
\tag{2.6}
$$

where for the last inequality we used Jensen's inequality.

Since

$$
\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt=\int_0^1 f\left(\|(1-t)\,y+tx\|^2\right)dt,
$$

hence by (2.6) we deduce (2.4). □

Lemma 1. Let f be continuous on $[0,\infty)$. Then for any $\lambda\in[0,1]$ and $x,y\in H$ we have the representation

$$
\begin{aligned}
&\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt\\
&=(1-\lambda)\int_0^1 f\left(\|(1-t)\left((1-\lambda)\,x+\lambda y\right)+ty\|^2\right)dt\\
&\quad+\lambda\int_0^1 f\left(\|(1-t)\,x+t\left((1-\lambda)\,x+\lambda y\right)\|^2\right)dt.
\end{aligned}
\tag{2.7}
$$

In particular, for $\lambda=\frac{1}{2}$,

$$
\begin{aligned}
\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt=&\frac{1}{2}\int_0^1 f\left(\left\|(1-t)\left(\frac{x+y}{2}\right)+ty\right\|^2\right)dt\\
&+\frac{1}{2}\int_0^1 f\left(\left\|(1-t)\,x+t\left(\frac{x+y}{2}\right)\right\|^2\right)dt.
\end{aligned}
\tag{2.8}
$$

Proof. For $\lambda=0$ and $\lambda=1$ the equality (2.7) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\begin{aligned}&\int_0^1 f\left(\|(1-t)(\lambda y+(1-\lambda)x)+ty\|^2\right)dt\\&=\int_0^1 f\left(\|((1-t)\lambda+t)y+(1-t)(1-\lambda)x\|^2\right)dt\end{aligned}$$

and

$$\int_0^1 f\left(\|t(\lambda y+(1-\lambda)x)+(1-t)x\|^2\right)dt=\int_0^1 f\left(\|t\lambda y+(1-\lambda t)x\|^2\right)dt.$$

If we make the change of variables $u:=(1-t)\lambda+t$, then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)du$. Then

$$\begin{aligned}&\int_0^1 f\left(\|((1-t)\lambda+t)y+(1-t)(1-\lambda)x\|^2\right)dt\\&=\frac{1}{1-\lambda}\int_\lambda^1 f\left(\|uy+(1-u)x\|^2\right)du.\end{aligned}$$

If we make the change of variables $u:=\lambda t$, then we have $du=\lambda dt$ and

$$\int_0^1 f\left(\|t\lambda y+(1-\lambda t)x\|^2\right)dt=\frac{1}{\lambda}\int_0^\lambda f\left(\|uy+(1-u)x\|^2\right)du.$$

Therefore

$$\begin{aligned}&(1-\lambda)\int_0^1 f\left(\|(1-t)(\lambda y+(1-\lambda)x)+ty\|^2\right)dt\\&\quad+\lambda\int_0^1 f\left(\|t(\lambda y+(1-\lambda)x)+(1-t)x\|^2\right)dt\\&=\int_\lambda^1 f\left(\|uy+(1-u)x\|^2\right)du\\&\quad+\int_0^\lambda f\left(\|uy+(1-u)x\|^2\right)du\\&=\int_0^1 f\left(\|uy+(1-u)x\|^2\right)du\end{aligned}$$

and the identity (2.7) is proved. □

Theorem 3. *Let $f : [0, \infty) \to \mathbb{R}$ be convex on $[0, \infty)$. If $x, y \in H$ with $\operatorname{Re}\langle x, y\rangle \geq 0$, then for $\lambda \in [0, 1]$*

$$\begin{aligned} &f\left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\right) \\ &\leq (1-\lambda) f\left(\frac{\|(1-\lambda)x + \lambda y\|^2 + (1-\lambda)\operatorname{Re}\langle x, y\rangle + (\lambda+1)\|y\|^2}{3}\right) \\ &\quad + \lambda f\left(\frac{(2-\lambda)\|x\|^2 + \lambda \operatorname{Re}\langle x, y\rangle + \|(1-\lambda)x + \lambda y\|^2}{3}\right) \\ &\leq \int_0^1 f\left(\|(1-t)x + ty\|^2\right) dt \qquad (2.9) \\ &\leq \frac{1}{3} f\left(\|(1-\lambda)x + \lambda y\|^2\right) + \frac{1}{3}\lambda f\left(\|x\|^2\right) + \frac{1}{3}(1-\lambda) f\left(\|y\|^2\right) \\ &\quad + \frac{1}{3}(1-\lambda) f\left((1-\lambda)\operatorname{Re}\langle x, y\rangle + \lambda\|y\|^2\right) \\ &\quad + \frac{1}{3}\lambda f\left((1-\lambda)x + \lambda \operatorname{Re}\langle x, y\rangle\right) \\ &\leq \frac{1}{3}\left[f\left(\|x\|^2\right) + f\left(\operatorname{Re}\langle x, y\rangle\right) + f\left(\|y\|^2\right)\right]. \end{aligned}$$

In particular,

$$\begin{aligned} &f\left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\right) \\ &\leq \frac{1}{2} f\left(\frac{1}{3}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{1}{2}\operatorname{Re}\langle x, y\rangle + \frac{3}{2}\|y\|^2\right]\right) \\ &\quad + \frac{1}{2} f\left(\frac{1}{3}\left[\frac{3}{2}\|x\|^2 + \frac{1}{2}\operatorname{Re}\langle x, y\rangle + \left\|\frac{x+y}{2}\right\|^2\right]\right) \\ &\leq \int_0^1 f\left[\|(1-t)x + ty\|^2\right] dt \qquad (2.10) \\ &\leq \frac{1}{3} f\left(\left\|\frac{x+y}{2}\right\|^2\right) + \frac{1}{6} f\left(\|x\|^2\right) + \frac{1}{6} f\left(\|y\|^2\right) \\ &\quad + \frac{1}{6} f\left[\frac{\operatorname{Re}\langle x, y\rangle + \|y\|^2}{2}\right] + \frac{1}{6} f\left[\frac{x + \operatorname{Re}\langle x, y\rangle}{2}\right] \\ &\leq \frac{1}{3}\left[f\left(\|x\|^2\right) + f\left(\operatorname{Re}\langle x, y\rangle\right) + f\left(\|y\|^2\right)\right]. \end{aligned}$$

If f is operator concave on $[0, \infty)$, then the sign of inequality reverses in (2.9) and (2.10).

Proof. By using (2.1) and replacing x with $(1-\lambda)x+\lambda y$, we get

$$
\begin{aligned}
& f\left(\frac{\|(1-\lambda)x+\lambda y\|^2+\operatorname{Re}\langle y,(1-\lambda)x+\lambda y\rangle+\|y\|^2}{3}\right) \\
& \leq \int_0^1 f\left[|(1-t)((1-\lambda)x+\lambda y)+ty|^2\right]dt \\
& \leq \frac{1}{3}\left[f\left(\|(1-\lambda)x+\lambda y\|^2\right)+f\left[\operatorname{Re}\langle y,(1-\lambda)x+\lambda y\rangle\right]+f\left(\|y\|^2\right)\right],
\end{aligned}
$$

which, by multiplication with $(1-\lambda)$ gives

$$
\begin{aligned}
& (1-\lambda) f\left(\frac{\|(1-\lambda)x+\lambda y\|^2+(1-\lambda)\operatorname{Re}\langle x,y\rangle+(\lambda+1)\|y\|^2}{3}\right) \\
& \leq (1-\lambda)\int_0^1 f\left(\|(1-t)((1-\lambda)x+\lambda y)+ty\|^2\right)dt \\
& \leq \frac{1}{3}\left[(1-\lambda) f\left(\|(1-\lambda)x+\lambda y\|^2\right)\right. \\
& \qquad \left. + f\left((1-\lambda)\operatorname{Re}\langle x,y\rangle+\lambda\|y\|^2\right)+f\left(\|y\|^2\right)\right].
\end{aligned} \tag{2.11}
$$

By using (2.1) and replacing y with $(1-\lambda)x+\lambda y$, we get

$$
\begin{aligned}
& f\left(\frac{\|x\|^2+\operatorname{Re}\langle(1-\lambda)x+\lambda y,x\rangle+\|(1-\lambda)x+\lambda y\|^2}{3}\right) \\
& \leq \int_0^1 f\left(\|(1-t)x+t((1-\lambda)x+\lambda y)\|^2\right)dt \\
& \leq \frac{1}{3}\left[f\left(\|x\|^2\right)+f\left[\operatorname{Re}\langle((1-\lambda)x+\lambda y),x\rangle\right]+f\left(\|(1-\lambda)x+\lambda y\|^2\right)\right],
\end{aligned}
$$

which, by multiplication with λ gives

$$
\begin{aligned}
& \lambda f\left(\frac{(2-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle+\|(1-\lambda)x+\lambda y\|^2}{3}\right) \\
& \leq \lambda\int_0^1 f\left(\|(1-t)x+t((1-\lambda)x+\lambda y)\|^2\right)dt \\
& \leq \frac{1}{3}\lambda\left[f\left(\|x\|^2\right)+f\left((1-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle\right)\right. \\
& \qquad \left. + f\left(\|(1-\lambda)x+\lambda y\|^2\right)\right].
\end{aligned} \tag{2.12}
$$

If we add (2.11) and (2.12), and use (2.7), then we get

$$
\begin{aligned}
&(1-\lambda) f\left(\frac{\|(1-\lambda)x+\lambda y\|^2+(1-\lambda)\operatorname{Re}\langle x,y\rangle+(\lambda+1)\|y\|^2}{3}\right)\\
&\quad+\lambda f\left(\frac{(2-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle+\|(1-\lambda)x+\lambda y\|^2}{3}\right)\\
&\le \int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt\\
&\le \frac{1}{3}(1-\lambda)\left[f\left(\|(1-\lambda)x+\lambda y\|^2\right)+f\left[(1-\lambda)\operatorname{Re}\langle x,y\rangle+\lambda\|y\|^2\right]+f\left(\|y\|^2\right)\right]\\
&\quad+\frac{1}{3}\lambda\left[f\left(\|x\|^2\right)+f\left[(1-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle\right]+f\left(\|(1-\lambda)x+\lambda y\|^2\right)\right]\\
&=\frac{1}{3}f\left(\|(1-\lambda)x+\lambda y\|^2\right)+\frac{1}{3}\lambda f\left(\|x\|^2\right)+\frac{1}{3}(1-\lambda)f\left(\|y\|^2\right)\\
&\quad+\frac{1}{3}(1-\lambda)f\left[(1-\lambda)\operatorname{Re}\langle x,y\rangle+\lambda\|y\|^2\right]+\frac{1}{3}\lambda f\left[(1-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle\right]\\
&=\frac{1}{3}f\left((1-\lambda)^2\|x\|^2+2\lambda(1-\lambda)\operatorname{Re}\langle x,y\rangle+\lambda\|y\|^2\right)\\
&\quad+\frac{1}{3}\lambda f\left(\|x\|^2\right)+\frac{1}{3}(1-\lambda)f\left(\|y\|^2\right)\\
&\quad+\frac{1}{3}(1-\lambda)f\left[(1-\lambda)\operatorname{Re}\langle x,y\rangle+\lambda\|y\|^2\right]+\frac{1}{3}\lambda f\left[(1-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle\right],
\end{aligned}
$$

which proves the second, third and fourth inequalities in (2.9).

By the operator convexity of f we have

$$
\begin{aligned}
&(1-\lambda) f\left(\frac{\|(1-\lambda)x+\lambda y\|^2+(1-\lambda)\operatorname{Re}\langle x,y\rangle+(\lambda+1)\|y\|^2}{3}\right)\\
&\quad+\lambda f\left(\frac{(2-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle+\|(1-\lambda)x+\lambda y\|^2}{3}\right)\\
&\ge f\left[(1-\lambda)\frac{\|(1-\lambda)x+\lambda y\|^2+(1-\lambda)\operatorname{Re}\langle x,y\rangle+(\lambda+1)\|y\|^2}{3}\right.\\
&\quad\left.+\lambda\frac{(2-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle+\|(1-\lambda)x+\lambda y\|^2}{3}\right]\\
&=f\left[\frac{1}{3}\left(\|(1-\lambda)x+\lambda y\|^2+\left[(1-\lambda)^2+\lambda^2\right]\operatorname{Re}\langle x,y\rangle\right.\right.\\
&\quad\left.\left.+\left(1-\lambda^2\right)\|y\|^2+(2-\lambda)\lambda\|x\|^2\right)\right]
\end{aligned}
$$

$$= f\left[\frac{1}{3}\Big((1-\lambda)^2\|x\|^2 + 2(1-\lambda)\lambda\operatorname{Re}\langle x,y\rangle + \lambda^2\|y\|^2\right.$$
$$\left. + \left[(1-\lambda)^2+\lambda^2\right]\operatorname{Re}\langle x,y\rangle + \left(1-\lambda^2\right)\|y\|^2 + (2-\lambda)\lambda\|x\|^2\Big)\right]$$
$$= f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right),$$

which proves the first inequality in (2.9).

By the operator convexity, we also have

$$\frac{1}{3}f\left((1-\lambda)^2\|x\|^2+2\lambda(1-\lambda)\operatorname{Re}\langle x,y\rangle+\lambda^2\|y\|^2\right)$$
$$+\frac{1}{3}\lambda f\left(\|x\|^2\right)+\frac{1}{3}(1-\lambda)f\left(\|y\|^2\right)$$
$$+\frac{1}{3}(1-\lambda)f\left[(1-\lambda)\operatorname{Re}\langle x,y\rangle+\lambda\|y\|^2\right]+\frac{1}{3}\lambda f\left[(1-\lambda)\|x\|^2+\lambda\operatorname{Re}\langle x,y\rangle\right]$$
$$\le\frac{1}{3}(1-\lambda)^2f\left(\|x\|^2\right)+\frac{2}{3}\lambda(1-\lambda)f(\operatorname{Re}\langle x,y\rangle)+\frac{1}{3}\lambda^2f\left(\|y\|^2\right)$$
$$+\frac{1}{3}\lambda f\left(\|x\|^2\right)+\frac{1}{3}(1-\lambda)f\left(\|y\|^2\right)$$
$$+\frac{1}{3}(1-\lambda)^2f(\operatorname{Re}\langle x,y\rangle)+\frac{1}{3}(1-\lambda)\lambda f\left(\|y\|^2\right)$$
$$+\frac{1}{3}\lambda(1-\lambda)f\left(\|x\|^2\right)+\lambda^2f(\operatorname{Re}\langle x,y\rangle)$$
$$=\frac{1}{3}\left[(1-\lambda)^2+\lambda+\lambda(1-\lambda)\right]f\left(\|x\|^2\right)$$
$$+\frac{1}{3}\left[2\lambda(1-\lambda)+(1-\lambda)^2+\lambda^2\right]f(\operatorname{Re}\langle x,y\rangle)$$
$$+\frac{1}{3}\left[\lambda^2+(1-\lambda)+(1-\lambda)\lambda\right]f\left(\|y\|^2\right)$$
$$=\frac{1}{3}\left[f\left(\|x\|^2\right)+f(\operatorname{Re}\langle x,y\rangle)+f\left(\|y\|^2\right)\right],$$

which proves the last part of (2.9). □

3. Some examples. Assume that $x,y\in H$ with $\operatorname{Re}\langle x,y\rangle\ge 0$, then by writing inequality (2.1) for the convex function $f(t)=t^r$ for $r\in[1,\infty)$, we get

$$\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^r\le\int_0^1\|(1-t)x+ty\|^{2r}dt$$
$$\le\frac{1}{3}\left[\|x\|^{2r}+[\operatorname{Re}\langle x,y\rangle]^r+\|y\|^{2r}\right]. \tag{3.1}$$

If we take the power $\frac{1}{2r}$, then

$$\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^{1/2} \le \left(\int_0^1 \|(1-t)\,x+ty\|^{2r}\,dt\right)^{1/2r} \le \frac{1}{3^{1/2r}}\left[\|x\|^{2r}+[\operatorname{Re}\langle x,y\rangle]^r+\|y\|^{2r}\right]^{1/2r}. \tag{3.2}$$

If $p\in(0,1)$, then by (2.1) for the concave function $f(t)=t^p$ we get

$$\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^{p} \ge \int_0^1 \|(1-t)\,x+ty\|^{2p}\,dt \ge \frac{1}{3}\left[\|x\|^{2p}+[\operatorname{Re}\langle x,y\rangle]^p+\|y\|^{2p}\right]. \tag{3.3}$$

In particular, for $p=1/2$, we get

$$\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^{1/2} \ge \int_0^1 \|(1-t)\,x+ty\|\,dt \ge \frac{1}{3}\left[\|x\|+[\operatorname{Re}\langle x,y\rangle]^{1/2}+\|y\|\right] \tag{3.4}$$

for x, $y\in H$ with $\operatorname{Re}\langle x,y\rangle\ge 0$.

If $\|x\|^2$, $\|y\|^2$, $\operatorname{Re}\langle x,y\rangle>0$, then for the concave function $f(t)=\ln t$ on $(0,\infty)$ we have

$$\begin{aligned}&\ln\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\ &\ge \int_0^1 \ln\left(\|(1-t)\,x+ty\|^2\right)dt\\ &\ge \frac{1}{3}\left[\ln\left(\|x\|^2\right)+\ln\left[\operatorname{Re}\langle x,y\rangle\right]+\ln\left(\|y\|^2\right)\right].\end{aligned} \tag{3.5}$$

If $\|x\|^2$, $\|y\|^2$, $\operatorname{Re}\langle x,y\rangle>0$, then for the convex function $f(t)=t^{-p}$ on $(0,\infty)$ with $p\in(0,\infty)$ we have

$$\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^{-p} \le \int_0^1 \|(1-t)\,x+ty\|^{-2p}\,dt \le \frac{1}{3}\left[\|x\|^{-2p}+[\operatorname{Re}\langle x,y\rangle]^{-p}+\|y\|^{-2p}\right] \tag{3.6}$$

and, in particular,

$$\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^{-1} \le \int_0^1 \|(1-t)\,x+ty\|^{-2}\,dt \le \frac{1}{3}\left[\|x\|^{-2}+[\operatorname{Re}\langle x,y\rangle]^{-1}+\|y\|^{-2}\right]. \tag{3.7}$$

Also, if we take $p = 1/2$ in (3.6), we get

$$\left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\right)^{-1/2} \le \int_0^1 \|(1-t)\,x + ty\|^{-1}\,dt \le \frac{1}{3}\left[\|x\|^{-1} + [\operatorname{Re}\langle x, y\rangle]^{-1/2} + \|y\|^{-1}\right]. \tag{3.8}$$

If $\|x\|^2$, $\|y\|^2$, $\operatorname{Re}\langle x, y\rangle > 0$, then for the convex function $f(t) = t\ln t$ on $(0,\infty)$ we have

$$\begin{aligned}&\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\ln\left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\right)\\ &\le \int_0^1 \|(1-t)\,x + ty\|^2 \ln\left(\|(1-t)\,x + ty\|^2\right) dt\\ &\le \frac{1}{3}\left[\|x\|^2\ln\left(\|x\|^2\right) + \operatorname{Re}\langle x, y\rangle \ln\left[\operatorname{Re}\langle x, y\rangle\right] + \|y\|^2\ln\left(\|y\|^2\right)\right].\end{aligned} \tag{3.9}$$

4. The case of real numbers. Assume that $f : [0,\infty) \to \mathbb{R}$ is convex (concave) and $0 \le a, b$. Then by (2.1) we get

$$f\left(\frac{a^2 + ab + b^2}{3}\right) \le (\ge) \int_0^1 f\left(((1-t)\,a + ta)^2\right) dt \le (\ge) \frac{1}{3}\left[f\left(a^2\right) + f(ab) + f\left(b^2\right)\right]. \tag{4.1}$$

By the change of variables $(1-t)\,a + ta = x$ in (4.1) we get

$$f\left(\frac{a^2 + ab + b^2}{3}\right) \le (\ge) \frac{1}{b-a}\int_a^b f\left(x^2\right) dx \le (\ge) \frac{1}{3}\left[f\left(a^2\right) + f(ab) + f\left(b^2\right)\right]. \tag{4.2}$$

We recall the following special means:

a) The *arithmetic mean*

$$A(a, b) := \frac{a+b}{2}, \quad a, b > 0.$$

b) The *geometric mean*

$$G(a, b) := \sqrt{ab}, \quad a, b \ge 0.$$

c) The *harmonic mean*

$$H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0.$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a,b>0,\ b\neq a, \\ a & \text{if } b=a>0. \end{cases}$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a,b>0,\ b\neq a, \\ a & \text{if } b=a>0. \end{cases}$$

f) The *p-logarithmic mean* ($p \in \mathbb{R}\setminus\{-1,0\}$)

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if } a,b>0,\ b\neq a, \\ a & \text{if } b=a>0. \end{cases}$$

It is well known that, if $L_{-1} := L$ and $L_0 := I$, then the function $\mathbb{R} \ni p \mapsto L_p$ is monotonically strictly increasing. In particular, we have

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b).$$

Let $p \geq 1$, then $f(x) = x^p$ is convex on $[0,\infty)$ and for $a,b \in [0,\infty)$ with $a \neq b$ from (4.2) we get

$$\sqrt{\frac{a^2+ab+b^2}{3}} \leq L_{2p}(a,b) \leq \left(\frac{a^{2p}+a^pb^p+b^{2p}}{3}\right)^{\frac{1}{2p}}. \tag{4.3}$$

For $q \in (0,1)$ the function $f(x) = x^q$ is concave on $[0,\infty)$ and for $a,\ b \in [0,\infty)$ with $a \neq b$ from (4.2) we get

$$L_{2q}(a,b) \leq \sqrt{\frac{a^2+ab+b^2}{3}} \leq \left(\frac{a^{2q}+a^qb^q+b^{2q}}{3}\right)^{\frac{1}{2q}}. \tag{4.4}$$

For $r \in (-\infty,-1)\cup(-1,0)$ the function $f(x) = x^r$ is convex on $(0,\infty)$ and for $a,b \in (0,\infty)$ with $a \neq b$ from (4.2) we get

$$\sqrt{\frac{a^2+ab+b^2}{3}} \geq L_{2r}(a,b) \geq \left(\frac{a^{2r}+a^rb^r+b^{2r}}{3}\right)^{\frac{1}{2r}}, \tag{4.5}$$

provided that $r \neq -\frac{1}{2}$.

If $r = -\frac{1}{2}$, then we get

$$\sqrt{\frac{a^2+ab+b^2}{3}} \geq L_{-1}^{-1}(a,b) \geq \left(\frac{a^{-1}+a^{-1/2}b^{-1/2}+b^{-1}}{3}\right)^{-1}. \tag{4.6}$$

For $\alpha \in \mathbb{R}$ we consider the convex function $f(x) = \exp(\alpha x)$. By (4.2) we derive

$$\exp\left(\alpha\frac{a^2+ab+b^2}{3}\right) \leq \frac{1}{b-a}\int_a^b \exp\left(\alpha x^2\right) dx \leq \frac{1}{3}\left[\exp\left(\alpha a^2\right) + \exp(\alpha ab) + \exp\left(\alpha b^2\right)\right] \tag{4.7}$$

for real numbers a, b with $a \neq b$.

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