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On naturality of some construction of connections

Dedicated to Professor Ivan Kolář on the occasion of his 85-th birthday.

Abstract. Let F be a bundle functor on the category of all fibred manifolds and fibred maps. Let Γ be a general connection in a fibred manifold $\mathrm{pr} : Y \to M$ and ∇ be a classical linear connection on M. We prove that the well-known general connection $\mathcal{F}(\Gamma, \nabla)$ in $FY \to M$ is canonical with respect to fibred maps and with respect to natural transformations of bundle functors.

Introduction. We assume that any manifold considered in the paper is Hausdorff, second countable, finite dimensional, without boundary and smooth (i.e. of class C^∞). All maps between manifolds are assumed to be smooth (of class C^∞). A general connection in a fibred manifold $\mathrm{pr} : Y \to M$ is a map

$$\Gamma : TM \times_M Y \to TY$$

such that

$$\Gamma(-, y) : T_x M \to T_y Y \text{ is linear and } T_y\mathrm{pr} \circ \Gamma(-, y) = \mathrm{id}_{T_x M}$$

for any $y \in Y_x$ and $x \in M$.

General connections Γ and Γ_1 in fibred manifolds $\mathrm{pr} : Y \to M$ and $\mathrm{pr}_1 : Y_1 \to M_1$ (respectively) are called to be f-related with respect to

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a fibred map $f : Y \to Y_1$ with the base map $\underline{f} : M \to M_1$ if

$$Tf \circ \Gamma(v, y) = \Gamma_1(T\underline{f}(v), f(y))$$

for any $v \in T_x M$, $y \in Y_x$ and $x \in M$.

A classical linear connection on a manifold M is a general connection ∇ in the tangent bundle $TM \to M$ of M such that ∇ and ∇ are a_t-related for any $t \in \mathbf{R}_+$, where $a_t : TM \to TM$ is the fiber multiplication by t. It is well known that such ∇ defines a linear connection

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

in the usual sense of [1] (and vice versa). One can see that if classical linear connections ∇ on M and ∇^1 on M_1 are f-related (i.e. Tf-related) for a map $f : M \to M_1$, then $\nabla_X Z$ and $\nabla^1_{X_1} Z_1$ are f-related if X and X_1 are and Z and Z_1 are.

We have the well-known canonical constructions on connections.

Example 0.1. Let ∇ be a classical linear connection on a manifold M and let $v \in T_{x_o}M$ be a vector tangent to M at a point $x_o \in M$. Denote by $\hat{v}$ the constant vector field on $T_{x_o}M$ determined by v, i.e. $\hat{v}(w) := \frac{d}{d\tau}_{|0}(w + \tau v)$, $w \in T_{x_o}M$. Then on some neighborhood of x_o we have the vector field

$$v^{[\nabla, x_o]} := (\mathcal{E}xp_{\nabla, x_o})_* \hat{v} \ , \tag{1}$$

the image of $\hat{v}$ by the geodesic exponent $\mathcal{E}xp_{\nabla, x_o} : (T_{x_o}M, 0) \to (M, x_o)$ of ∇ at x_o.

Example 0.2. Let Γ be a general connection in a fibred manifold $\mathrm{pr} : Y \to M$ and ∇ be a classical linear connection on M. Let $y_o \in Y_{x_o}$, $x_o \in M$. Let $v \in T_{x_o}M$. Then on some neighborhood of y_o we have the projectable vector field

$$v^{[\Gamma, \nabla, y_o]} := \left(v^{[\nabla, x_o]}\right)^\Gamma , \tag{2}$$

where $X^\Gamma = \Gamma(X, -)$ is the Γ-horizontal lift of a vector field X on M to Y.

Example 0.3. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor on the category $\mathcal{FM}_{m,n}$ of fibred manifolds with n-dimensional fibres and m-dimensional bases and (locally defined) fibred diffeomorphisms. Let $\Gamma : TM \times_M Y \to TY$ be a general connection in a $\mathcal{FM}_{m,n}$-object $\mathrm{pr} : Y \to M$ and ∇ be a classical linear connection on M. Then we have a map $\mathcal{F}(\Gamma, \nabla) : TM \times_M FY \to TFY$ defined by

$$\mathcal{F}(\Gamma, \nabla)(v, z) := \mathcal{F}v^{[\Gamma, \nabla, y_o]}(z) \, , \quad z \in F_{y_o}Y \ , \ v \in T_{x_o}M \ , \ y_o \in Y_{x_o} \ , \ x_o \in M \ ,$$

where $\mathcal{F}X$ denotes the flow lift of a projectable vector field X in $Y \to M$ to FY by means of F. Then $\mathcal{F}(\Gamma, \nabla)$ is a general connection in $FY \to M$. One can see that it is the composition of $\mathcal{F}(\Gamma, \Lambda)$ from Item 45.4 in [2] with exponential extension of ∇ into r-th order linear connection $\Lambda(\nabla)$.

Clearly, the construction of $\mathcal{F}(\Gamma, \nabla)$ is $\mathcal{FM}_{m,n}$-canonical, i.e. we have the corresponding $\mathcal{FM}_{m,n}$-natural operator in the sense of [2]. More precisely, we have:

Proposition 0.4. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. Let $\mathrm{pr} : Y \to M$ and $\mathrm{pr}_1 : Y_1 \to M_1$ be $\mathcal{FM}_{m,n}$-objects. Let $f : Y \to Y_1$ be a (locally defined) fibred diffeomorphism with the base map $\underline{f} : M \to M_1$. Let $\check{\nabla}$ be a classical linear connection on M and $\breve{\nabla}$ be a classical linear connection on M_1. Assume that $\check{\nabla}$ and $\breve{\nabla}$ are $\underline{f}$-related. Let $\check{\Gamma}$ be a general connection in $\mathrm{pr} : Y \to M$ and $\breve{\Gamma}$ be a general connection in $\mathrm{pr}_1 : Y_1 \to M_1$. Assume that connections $\check{\Gamma}$ and $\breve{\Gamma}$ are f-related. Then the general connections $\mathcal{F}(\check{\Gamma}, \check{\nabla})$ and $\mathcal{F}(\breve{\Gamma}, \breve{\nabla})$ are Ff-related.*

The purpose of the note is to prove that given a bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ on the category $\mathcal{FM}$ of all fibred manifolds and fibred maps, the construction of $\mathcal{F}(\Gamma, \nabla)$ is $\mathcal{FM}$-canonical. More precisely, we will prove:

Theorem 0.5. *Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor. Let $\mathrm{pr} : Y \to M$ and $\mathrm{pr}_1 : Y_1 \to M_1$ be fibred manifolds. Let $f : Y \to Y_1$ be a fibred map with the base map $\underline{f} : M \to M_1$. Let $\check{\nabla}$ be a classical linear connection on M and $\breve{\nabla}$ be a classical linear connection on M_1. Assume that $\check{\nabla}$ and $\breve{\nabla}$ are $\underline{f}$-related. Let $\check{\Gamma}$ be a general connection in $\mathrm{pr} : Y \to M$ and $\breve{\Gamma}$ be a general connection in $\mathrm{pr}_1 : Y_1 \to M_1$. Assume that connections $\check{\Gamma}$ and $\breve{\Gamma}$ are f-related. Then the general connections $\mathcal{F}(\check{\Gamma}, \check{\nabla})$ and $\mathcal{F}(\breve{\Gamma}, \breve{\nabla})$ are Ff-related.*

We also deduce that the construction of $\mathcal{F}(\Gamma, \nabla)$ is canonical with respect to F. More precisely, we will prove:

Theorem 0.6. *Let $F, F_1 : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be bundle functors and $\mu : F \to F_1$ be a $\mathcal{FM}_{m,n}$-natural transformation. Let $\mathrm{pr} : Y \to M$ be a $\mathcal{FM}_{m,n}$-object. Let $\check{\nabla}$ be a classical linear connection on M. Let $\check{\Gamma}$ be a general connection in $\mathrm{pr} : Y \to M$. Then the general connections $\mathcal{F}(\check{\Gamma}, \check{\nabla})$ and $\mathcal{F}_1(\check{\Gamma}, \check{\nabla})$ are μ_Y-related.*

1. Some preparatory lemmas.

Lemma 1.1. *Let m, m_1 be non-negative integers and p be an integer such that $0 \leq p \leq \min\{m, m_1\}$. Let $v = (v^1, \dots, v^m) \in T_0\mathbf{R}^m = \mathbf{R}^m$ be a vector. Let $\iota : \mathbf{R}^m \to \mathbf{R}^{m_1}$ be given by $\iota(x^1, \dots, x^m) = (x^1, \dots, x^p, 0, \dots, 0)$. Let $\check{\nabla}$ be a classical linear connection on $\mathbf{R}^m$ and $\breve{\nabla}$ be a classical linear connection on $\mathbf{R}^{m_1}$. Assume that $\check{\nabla}$ and $\breve{\nabla}$ are ι-related. Suppose $\gamma = (\gamma^1, \dots, \gamma^m)$ is the $\check{\nabla}$-geodesic such that $\gamma(0) = 0$ and $\gamma'(0) = v = (v^1, \dots, v^m)$. Then $\breve{\gamma} := \iota \circ \gamma = (\gamma^1, \dots, \gamma^p, 0, \dots, 0)$ is the $\breve{\nabla}$-geodesic such that $\breve{\gamma}(0) = \iota(0)$ and $\breve{\gamma}'(0) = T\iota(v) = (v^1, \dots, v^p, 0, \dots, 0)$.*

Proof. If $m=0$ or $m_1=0$ or $p=0$, then $\iota=0$. Then $\breve{\gamma}=0$, and then it is $\breve{\nabla}$-geodesic. So, we may additionally assume that m,m_1,p are positive integers. Let $\breve{\Gamma}^\rho_{\alpha\beta}$ be the Christofell symbols of $\breve{\nabla}$ with respect to the usual coordinates on $\mathbf{R}^m$ and $\breve{\Gamma}^s_{qr}$ be the Christofell symbols of $\breve{\nabla}$ with respect to the usual coordinates on $\mathbf{R}^{m_1}$. Since $\breve{\nabla}$ and $\breve{\nabla}$ are ι-related, then:

$$
\begin{aligned}
&\breve{\Gamma}^s_{ij}(x^1,...,x^p,0,...,0)=0 \text{ for } i,j=1,...,p \text{ and } s=p+1,...,m_1\,;\\
&\breve{\Gamma}^k_{ij}(x^1,...,x^m)=\breve{\Gamma}^k_{ij}(x^1,...,x^p,0,...,0) \text{ for } i,j,k=1,...,p\,;\\
&\breve{\Gamma}^k_{qr}(x^1,...,x^m)=0 \text{ for } k=1,...,p\,,\ q=p+1,...,m\,, r=1,...,m\,;\\
&\breve{\Gamma}^k_{qr}(x^1,...,x^m)=0 \text{ for } k=1,...,p\,,\ q=1,...,m\,,\ r=p+1,...,m\,.
\end{aligned}
\tag{3}
$$

Indeed, we can see that $T\iota\circ\partial\rho=\partial_\rho\circ\iota$ for $\rho=1,\dots,p$ and $=0$ for $\rho=p+1,\dots,m$. Then

$$T\iota((\breve{\nabla}_{\partial_\alpha}\partial_\beta)_{|(x^1,\dots,x^m)})=\sum_{\rho=1}^{p}\breve{\Gamma}^\rho_{\alpha\beta}(x^1,\dots,x^m)\partial_{\rho|(x^1,\dots,x^p,0,\dots,0)}$$

and (since $\breve{\nabla}$ and $\breve{\nabla}$ are ι-related)

$$
\begin{aligned}
T\iota((\breve{\nabla}_{\partial_\alpha}\partial_\beta)_{|(x^1,\dots,x^m)})&=\breve{\nabla}_{\partial_\alpha}\partial_{\beta|(x^1,\dots,x^p,0,\dots,0)}\\
&=\sum_{\rho=1}^{m_1}\breve{\Gamma}^\rho_{\alpha\beta}(x^1,\dots,x^p,0,\dots,0)\partial_{\rho|(x^1,\dots,x^p,0,\dots,0)}
\end{aligned}
$$

if $\alpha,\beta=1,\dots,p$ and $T\iota((\breve{\nabla}_{\partial_\alpha}\partial_\beta)_{|(x^1,\dots,x^m)})=0$ for other $\alpha,\beta=1,\dots,m$. Then considering the coefficients on $\partial_{\rho(x^1,\dots,x^p,0,\dots,0)}$, we get (3).

Since γ is a $\breve{\nabla}$-geodesic, then

$$\frac{d^2\gamma^\rho}{dt^2}=-\sum_{\alpha,\beta=1}^{m}\breve{\Gamma}^\rho_{\alpha_\beta}(\gamma)\frac{d\gamma^\alpha}{dt}\frac{d\gamma^\beta}{dt}\ ,\quad \rho=1,\dots,m\,.$$

Consequently, denoting $\breve{\gamma}=(\breve{\gamma}^1,\dots,\breve{\gamma}^{m_1})$, we get

$$\frac{d^2\breve{\gamma}^s}{dt^2}=-\sum_{q,r=1}^{m_1}\breve{\Gamma}^s_{qr}(\breve{\gamma})\frac{d\breve{\gamma}^q}{dt}\frac{d\breve{\gamma}^r}{dt}\quad\text{for } s=1,\dots,m_1\,.$$

Indeed, if $s=p+1,\dots,m_1$, then both sides of the above equations are equal to 0, and if $s=1,\dots,p$, then

$$
\begin{aligned}
\frac{d^2\breve{\gamma}^s}{dt^2}&=\frac{d^2\gamma^s}{dt^2}=-\sum_{\alpha,\beta=1}^{m}\breve{\Gamma}^s_{\alpha_\beta}(\gamma)\frac{d\gamma^\alpha}{dt}\frac{d\gamma^\beta}{dt}=-\sum_{q,r=1}^{p}\breve{\Gamma}^s_{qr}(\gamma)\frac{d\gamma^q}{dt}\frac{d\gamma^r}{dt}\\
&=-\sum_{q,r=1}^{p}\breve{\Gamma}^s_{qr}(\breve{\gamma})\frac{d\breve{\gamma}^q}{dt}\frac{d\breve{\gamma}^r}{dt}=-\sum_{q,r=1}^{m_1}\breve{\Gamma}^s_{qr}(\breve{\gamma})\frac{d\breve{\gamma}^q}{dt}\frac{d\breve{\gamma}^r}{dt}\,,
\end{aligned}
$$

as well. The lemma is complete. □

Lemma 1.2. *Let m, m_1 be non-negative integers and p be an integer such that $0 \le p \le \min\{m, m_1\}$. Let $v = (v^1, \dots, v^m) \in T_0\mathbf{R}^m = \mathbf{R}^m$ be a vector. Let $\iota : \mathbf{R}^m \to \mathbf{R}^{m_1}$ be given by $\iota(x^1, \dots, x^m) = (x^1, \dots, x^p, 0, \dots, 0)$. Let $\tilde{\nabla}$ be a classical linear connection on $\mathbf{R}^m$ and $\check{\nabla}$ be a classical linear connection on $\mathbf{R}^{m_1}$. Assume that $\tilde{\nabla}$ and $\check{\nabla}$ are ι related. Then the vector fields $v^{[\tilde{\nabla},0]}$ and $(T\iota(v))^{[\check{\nabla},\iota(0)]}$ are ι-related.*

Proof. Similarly as in Lemma 1.1, we may additionally assume that m, m_1, p are positive integers.

Consider a point $x \in \mathbf{R}^m$ near 0. Then $x = \mathcal{E}xp_{\tilde{\nabla},0}(w)$, where $w \in T_0\mathbf{R}^m$ is the point. Then

$$v^{[\tilde{\nabla},0]}_{|x} = \frac{d}{d\tau}_{|\tau=0}\gamma_\tau(1)\,,$$

where γ_τ is the $\tilde{\nabla}$-geodesic such that $\gamma_\tau(0) = 0$ and $\gamma'_\tau(0) = w + \tau v$ for any small $\tau \in \mathbf{R}$. Then (by Lemma 1.1) $\check{\gamma}_\tau := \iota \circ \gamma_\tau$ is the $\check{\nabla}$-geodesic such that $\check{\gamma}_\tau(0) = \iota(0)$ and $\check{\gamma}'_\tau(0) = T\iota(w + \tau v) = T\iota(w) + \tau T\iota(v)$. Hence

$$T\iota\big(v^{[\tilde{\nabla},0]}_{|x}\big) = T\iota\big(\tfrac{d}{d\tau}_{|\tau=0}\gamma_\tau(1)\big) = \tfrac{d}{d\tau}_{|\tau=0}\check{\gamma}_\tau(1) = (T\iota(v))^{[\check{\nabla},\iota(0)]}_{|\iota(x)}$$

for any small τ. The lemma is complete. □

Lemma 1.3. *Let m, m_1, n, n_1 be non-negative integers and p, q be integers such that $0 \le p \le \min\{m, m_1\}$ and $0 \le q \le \min\{n, n_1\}$. Let $v = (v^1, \dots, v^m) \in T_0\mathbf{R}^m = \mathbf{R}^m$ and $y_o = (0,0) \in (\mathbf{R}^m \times \mathbf{R}^n)_0 = \mathbf{R}^n$. Let $\iota : \mathbf{R}^m \to \mathbf{R}^{m_1}$ be given by $\iota(x^1, \dots, x^m) = (x^1, \dots, x^p, 0, \dots, 0)$ and $\kappa : \mathbf{R}^n \to \mathbf{R}^{n_1}$ be given by $\kappa(y^1, \dots, y^n) = (y^1, \dots, y^q, 0, \dots, 0)$. Let $\tilde{\nabla}$ be a classical linear connection on $\mathbf{R}^m$ and $\check{\nabla}$ be a classical linear connection on $\mathbf{R}^{m_1}$. Assume that $\tilde{\nabla}$ and $\check{\nabla}$ are ι related. Let $\tilde{\Gamma}$ be a general connection in the trivial bundle $\mathrm{pr} : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and $\check{\Gamma}$ be a general connection in the trivial bundle $\mathrm{pr}_1 : \mathbf{R}^{m_1} \times \mathbf{R}^{n_1} \to \mathbf{R}^{m_1}$. Assume that connections $\tilde{\Gamma}$ and $\check{\Gamma}$ are $(\iota \times \kappa, \iota)$-related. Then the vector fields $v^{[\tilde{\Gamma},\tilde{\nabla},y_o]}$ and $(T\iota(v))^{[\check{\Gamma},\check{\nabla},\iota\times\kappa(y_o)]}$ are $\iota \times \kappa$-related.*

Proof. Let $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$. Then $v^{[\tilde{\Gamma},\tilde{\nabla},y_o]}_{|(x,y)} \in \tilde{\Gamma}_{(x,y)}$. Then (since $\tilde{\Gamma}$ and $\check{\Gamma}$ are $(\iota \times \kappa, \iota)$-related)

$$T_{(x,y)}(\iota \times \kappa)\big(v^{[\tilde{\Gamma},\tilde{\nabla},y_o]}_{|(x,y)}\big) \in \check{\Gamma}_{(\iota(x),\kappa(y))}.$$

Moreover, using Lemma 1.2 and the property defining the $\tilde{\Gamma}$-horizontal lift, we get

$$\begin{aligned} T\mathrm{pr}_1 \circ T_{(x,y)}(\iota \times \kappa)\big(v^{[\tilde{\Gamma},\tilde{\nabla},y_o]}_{|(x,y)}\big) &= T\iota \circ T\mathrm{pr}\big(v^{[\tilde{\Gamma},\tilde{\nabla},y_o]}_{|(x,y)}\big) \\ &= T\iota\big(v^{[\tilde{\nabla},0]}_{|x}\big) = (T\iota(v))^{[\check{\nabla},\iota(0)]}_{|\iota(x)}\,. \end{aligned}$$

Hence

$$T(\iota\times\kappa)\circ v^{[\check{\Gamma},\check{\nabla},y_o]}_{|(x,y)} = ((T\iota(v))^{[\check{\nabla},\iota(0)]})^{\check{\Gamma}}_{|(\iota(x),\kappa(y))}$$
$$= (T\iota(v))^{[\check{\Gamma},\check{\nabla},\iota\times\kappa(y_o)]}\circ(\iota\times\kappa)(x,y)\,.$$

The lemma is complete. □

Lemma 1.4. *Let m, m_1 be non-negative integers and p be an integer such that $0\le p\le\min\{m,m_1\}$. Let $\iota:\mathbf{R}^m\to\mathbf{R}^{m_1}$ be given by $\iota(x^1,\dots,x^m) = (x^1,\dots,x^p,0,\dots,0)$. Let $X=\sum_{i=1}^m X^i\partial_i$ be a vector field on $\mathbf{R}^m$ and $X_1=\sum_{j=1}^{m_1}X_1^j\partial_j$ be a vector field on $\mathbf{R}^{m_1}$. Assume that X and X_1 are ι-related. Let $\{\varphi_t\}$ be the flow of X and $\{\psi_t\}$ be the flow of X_1. Then $\iota\circ\varphi_t=\psi_t\circ\iota$ for all sufficiently small t.*

Proof. We know that:

$$\frac{d}{dt}(\varphi_t^i(x^1,\dots,x^m)) = X^i(\varphi_t(x^1,\dots,x^m)) \text{ and } \varphi_0^i(x^1,\dots,x^m)=x^i$$

for $i=1,\dots,m$;

$$\frac{d}{dt}(\psi_t^j(x^1,\dots,x^{m_1})) = X_1^j(\psi_t(x^1,\dots,x^{m_1})) \text{ and } \psi_0^j(x^1,\dots,x^{m_1})=x^j$$

for $j=1,\dots,m_1$.

By the assumption that X and X_1 are ι-related, we have:

$$X^i(x^1,\dots,x^m) = X_1^i(x^1,\dots,x^p,0,\dots,0) \text{ for } i=1,\dots,p\,;$$
$$X_1^j(x^1,\dots,x^p,0,\dots,0)=0 \text{ for } j=p+1,\dots,m_1\,.$$

Then (because of the well-known uniqueness result of systems of ordinary differential equations) we derive:

$$\varphi_t^k(x^1,\dots,x^m)=\varphi_t^k(x^1,\dots,x^p,0,\dots,0)=\psi_t^k(x^1,\dots,x^p,0,\dots,0)$$

for $k=1,\dots,p$;

$$\psi_t^k(x^1,\dots,x^p,0,\dots,0)=0 \text{ for } k=p+1,\dots,m_1\,.$$

The lemma is complete. □

Lemma 1.5. *Let m, m_1, n, n_1 be non-negative integers and p, q be integers such that $0\le p\le\min\{m,m_1\}$ and $0\le q\le\min\{n,n_1\}$. Let $\iota:\mathbf{R}^m\to\mathbf{R}^{m_1}$ be given by $\iota(x^1,\dots,x^m)=(x^1,\dots,x^p,0,\dots,0)$ and $\kappa:\mathbf{R}^n\to\mathbf{R}^{n_1}$ be given by $\kappa(y^1,\dots,y^n)=(y^1,\dots,y^q,0,\dots,0)$. Let X be a vector field on $\mathbf{R}^m\times\mathbf{R}^n$ and X_1 be a vector field on $\mathbf{R}^{m_1}\times\mathbf{R}^{n_1}$. Assume that X and X_1 are $\iota\times\kappa$-related. Let $\{\varphi_t\}$ be the flow of X and $\{\psi_t\}$ be the flow of X_1. Then $(\iota\times\kappa)\circ\varphi_t=\psi_t\circ(\iota\times\kappa)$ for all sufficiently small t.*

Proof. This lemma is the obvious modification of Lemma 1.4 for $(m+n, m_1+n_1, p+q)$ playing the role of (m,m_1,p). The lemma is complete. □

Lemma 1.6. *Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor. Let m, m_1, n, n_1 be non-negative integers and p, q be integers such that $0 \le p \le \min\{m, m_1\}$ and $0 \le q \le \min\{n, n_1\}$. Let $\iota : \mathbf{R}^m \to \mathbf{R}^{m_1}$ be given by $\iota(x^1, \dots, x^m) = (x^1, \dots, x^p, 0, \dots, 0)$ and $\kappa : \mathbf{R}^n \to \mathbf{R}^{n_1}$ be given by $\kappa(y^1, \dots, y^n) = (y^1, \dots, y^q, 0, \dots, 0)$. Let $\check{\nabla}$ be a classical linear connection on $\mathbf{R}^m$ and $\check{\nabla}$ be a classical linear connection on $\mathbf{R}^{m_1}$. Assume that $\check{\nabla}$ and $\check{\nabla}$ are ι-related. Let $\check{\Gamma}$ be a general connection in the trivial bundle $\mathrm{pr} : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and $\check{\Gamma}$ be a general connection in the trivial bundle $\mathrm{pr}_1 : \mathbf{R}^{m_1} \times \mathbf{R}^{n_1} \to \mathbf{R}^{m_1}$. Assume that connections $\check{\Gamma}$ and $\check{\Gamma}$ are $\iota \times \kappa$-related. Let $y_o = (0, 0) \in \mathbf{R}^m \times \mathbf{R}^n$. Let $v \in T_0\mathbf{R}^m$ and $z \in F_{y_o}(\mathbf{R}^m \times \mathbf{R}^n)$. Then*

$$TF(\iota \times \kappa)(\mathcal{F}(\check{\Gamma}, \check{\nabla})(v, z)) = \mathcal{F}(\check{\Gamma}, \check{\nabla})(T\iota(v), F(\iota \times \kappa)(z)) .$$

Proof. By Lemma 1.3, vector fields $v^{[\check{\Gamma}, \check{\nabla}, y_o]}$ and $(T\iota(v))^{[\check{\Gamma}, \check{\nabla}, \iota\times\kappa(y_o)]}$ are $\iota \times \kappa$-related. Let $\{\varphi_t\}$ be the flow of $v^{[\check{\Gamma}, \check{\nabla}, y_o]}$ and $\{\psi_t\}$ be the flow of $(T\iota(v))^{[\check{\Gamma}, \check{\nabla}, \iota\times\kappa(y_o)]}$. By Lemma 1.5, $(\iota \times \kappa) \circ \varphi_t = \psi_t \circ (\iota \times \kappa)$ for all sufficiently small reals t. Then

$$\begin{aligned} TF(\iota \times \kappa)(\mathcal{F}(\check{\Gamma}, \check{\nabla})(v, z)) &= TF(\iota \times \kappa)\Big(\frac{d}{dt}_{|t=0} F\varphi_t(z)\Big) \\ &= \frac{d}{dt}_{|t=0} F((\iota \times \kappa) \circ \varphi_t)(z) \\ &= \frac{d}{dt}_{|t=0} F(\psi_t \circ (\iota \times \kappa))(z) \\ &= \frac{d}{dt}_{t=0} F(\psi_t)(F(\iota \times \kappa)(z)) \\ &= \mathcal{F}(\check{\Gamma}, \check{\nabla})(T\iota(v), F(\iota \times \kappa)(z)) . \end{aligned}$$

The lemma is complete. □

Lemma 1.7. *Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor. Let $\mathrm{pr} : Y \to M$ and $\mathrm{pr}_1 : Y_1 \to M_1$ be fibred manifolds. Let $f : Y \to Y_1$ be a fibred map with the base map $\underline{f} : M \to M_1$. Assume that f and $\underline{f}$ are of constant rank. Let $\check{\nabla}$ be a classical linear connection on M and $\check{\nabla}$ be a classical linear connection on M_1. Assume that $\check{\nabla}$ and $\check{\nabla}$ are $\underline{f}$-related. Let $\check{\Gamma}$ be a general connection in $\mathrm{pr} : Y \to M$ and $\check{\Gamma}$ be a general connection in $\mathrm{pr}_1 : Y_1 \to M_1$. Assume that connections $\check{\Gamma}$ and $\check{\Gamma}$ are f-related. Let $v \in T_{x_o}M$ and $z \in F_{y_o}Y$, $y_o \in Y_{x_o}$, $x_o \in M$. Then*

$$TFf(\mathcal{F}(\check{\Gamma}, \check{\nabla})(v, z)) = \mathcal{F}(\check{\Gamma}, \check{\nabla})(T\underline{f}(v), Ff(z)) .$$

Proof. The lemma is clear if f is a (locally defined) fibred diffeomorphism, see Proposition 0.4. Then (by the rank theorem) we can additionally assume that $\mathrm{pr} : Y = \mathbf{R}^m \times \mathbf{R}^n \to M = \mathbf{R}^m$, $\mathrm{pr}_1 : Y_1 = \mathbf{R}^{m_1} \times \mathbf{R}^{n_1} \to M_1 = \mathbf{R}^{m_1}$

are the trivial bundles, $y_o = (0,0) \in \mathbf{R}^m \times \mathbf{R}^n$, $x_o = 0 \in \mathbf{R}^m$ and $f = \iota \times \kappa$. Then the lemma immediately follows from Lemma 1.6. □

2. The construction of $\boldsymbol{\mathcal{F}(\Gamma,\nabla)}$ is canonical with respect to $\boldsymbol{\mathcal{FM}}$. We have the following theorem corresponding to Theorem 0.5.

Theorem 2.1. *Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor. Let* $\mathrm{pr} : Y \to M$ *and* $\mathrm{pr}_1 : Y_1 \to M_1$ *be fibred manifolds. Let $f : Y \to Y_1$ be a fibred map with the base map $\underline{f} : M \to M_1$. Let $\check{\nabla}$ be a classical linear connection on M and $\check{\nabla}$ be a classical linear connection on M_1. Assume that $\check{\nabla}$ and $\check{\nabla}$ are $\underline{f}$-related. Let $\check{\Gamma}$ be a general connection in* $\mathrm{pr} : Y \to M$ *and $\check{\Gamma}$ be a general connection in* $\mathrm{pr}_1 : Y_1 \to M_1$. *Assume that connections $\check{\Gamma}$ and $\check{\Gamma}$ are f-related. Then the general connections $\mathcal{F}(\check{\Gamma},\check{\nabla})$ and $\mathcal{F}(\check{\Gamma},\check{\nabla})$ are Ff-related.*

Proof. Let $v \in T_{x_o}M$ and $z \in F_{y_o}Y$, $y_o \in Y_{x_o}$, $x_o \in M$. There is a sequence $y_n \in Y_{x_n}$ with $x_n \in M$ such that $y_n \to y_o$ if $n \to \infty$, $x_n \to x_o$ if $n \to \infty$, f is of constant rank on some neighborhood of y_n and $\underline{f}$ is of constant rank on some neighborhood of x_n for $n = 1, 2, \ldots$. (We can define y_n as follows. Let $V_1, \ldots, V_n, \ldots$ be open neighborhoods of y_o such that $V_1 \supset V_2 \supset \ldots$ and $\bigcap V_n = \{y_o\}$. Let $\mathrm{rank}_y(f)$ denote the rank of T_yf. Let $\tilde{y}_n \in V_n$ be a point such that $\mathrm{rank}_{\tilde{y}_n}(f) \geq \mathrm{rank}_y(f)$ for all $y \in V_n$. Let $U_n \subset V_n$ be an open neighborhood of $\tilde{y}_n$ such that $f_{|U_n}$ is of constant rank $\mathrm{rank}_{\tilde{y}_n}(f)$. Let $x_n \in \mathrm{pr}(U_n)$ be such that $\mathrm{rank}_{x_n}(\underline{f}) \geq \mathrm{rank}_x(\underline{f})$ for all $x \in \mathrm{pr}(U_n)$. Then choose an arbitrary point $y_n \in Y_{x_n} \cap U_n$.) Moreover, there is a sequence $z_n \in F_{y_n}Y$ such that $z_n \to z$ and there is a sequence $v_n \in T_{x_n}M$ such that $v_n \to v$. By Lemma 1.7,

$$TFf(\mathcal{F}(\check{\Gamma},\check{\nabla})(v_n,z_n)) = \mathcal{F}(\check{\Gamma},\check{\nabla})(T\underline{f}(v_n),Ff(z_n))\,.$$

Putting $n \to \infty$, we get

$$TFf(\mathcal{F}(\check{\Gamma},\check{\nabla})(v,z)) = \mathcal{F}(\check{\Gamma},\check{\nabla})(T\underline{f}(v),Ff(z))\,.$$

The theorem is complete. □

3. The construction of $\boldsymbol{\mathcal{F}(\Gamma,\nabla)}$ is canonical with respect to $\boldsymbol{F}$. We have the following theorem corresponding to Theorem 0.6.

Theorem 3.1. *Let $F, F_1 : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be bundle functors and $\mu : F \to F_1$ be a $\mathcal{FM}_{m,n}$-natural transformation. Let* $\mathrm{pr} : Y \to M$ *be a $\mathcal{FM}_{m,n}$-object. Let $\check{\nabla}$ be a classical linear connection on M. Let $\check{\Gamma}$ be a general connection in* $\mathrm{pr} : Y \to M$. *Then the general connections $\mathcal{F}(\check{\Gamma},\check{\nabla})$ and $\mathcal{F}_1(\check{\Gamma},\check{\nabla})$ are μ_Y-related.*

Proof. Let $v \in T_{x_o}M$ and $z \in F_{y_o}Y$, $y_o \in Y_{x_o}$, $x_o \in M$.

Let $\{\varphi_t\}$ be the flow of $v^{[\check{\Gamma},\check{\nabla},y_o]}$. Since μ is a natural transformation, then

$$\mu_Y \circ F\varphi_t = F_1\varphi_t \circ \mu_Y\,.$$

That is why $T\mu_Y \circ \mathcal{F}(\check{\Gamma}, \check{\nabla})(v, z) = \mathcal{F}_1(\check{\Gamma}, \check{\nabla})(v, \mu_Y(z))$. The theorem is complete. □

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